Review for Exam 3.

(Sections 11.2 – 11.10)

11.2 The series $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$,

if $|r| < 1$.

Note a = first term

Ex. Find $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{5^n}$.

$n=1 \rightarrow$ first term $= \dfrac{(-2)^2}{5}$

$r = \dfrac{a_2}{a_1} = \left(-\dfrac{2}{5}\right)$ = shrinking factor

$\therefore S = \dfrac{\frac{4}{5}}{1-\left(-\frac{2}{5}\right)} = \dfrac{4/5}{7/5} = \dfrac{4}{7}$

If $\sum_{n=0}^{\infty} a_n$ converges,

then $\lim_{n\to\infty} a_n = 0$.

But although $\lim_{n\to\infty} \frac{1}{n} = 0$,

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Ex. Does $\sum_{n=1}^{\infty} \frac{(-1)^n n \tan^{-1} n}{2n+3}$ converge?

Note that $\lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}$.

and $\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$

$\therefore \lim_{n\to\infty} \left| \frac{(-1)^n n \cdot \tan^{-1} n}{2n+3} \right|$

$= 1 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \neq 0$

Diverges since $\{a_n\}$ does <u>not</u> approach 0.

11.3 Integral Test. and

If f is continuous and

positive and decreasing,

and $a_n = f(n)$, then

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ is convergent, if } \int_a^{\infty} f(x)\,dx \text{ is convergent}}$$

and

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ is divergent if } \int_a^{\infty} f(x)\,dx \text{ is divergent.}}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

and diverges if $p \leq 1$.

Ex. $\sum_{n=0}^{\infty} n \cdot 2^{-n^2}$

$\int_0^T x \cdot 2^{-x^2}\, dx$ Set $u = x^2$

$\rightarrow du = 2x\, dx$

$\rightarrow x\, dx = \frac{du}{2}$

$\int_0^{T^2} 2^{-u} \cdot \frac{du}{2}$

$\frac{-1}{2\ln 2} \cdot 2^{-u} \Big|_0^{T^2} = \frac{-1}{2\ln 2}\left(2^{-T^2} - 1\right)$

$\to \dfrac{1}{2\ln 2}$ as $T \to \infty$

$\therefore \displaystyle\sum_{n=1}^{\infty} n \cdot 2^{-n^2}$ converges.

11.4 Comparison Test

Suppose a_n and $b_n > 0$ and $a_n \leq b_n$. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Also, if $a_n \geq b_n$, and

$\sum_{n=1}^{\infty} b_n$ diverges, then

so does $\sum_{n=1}^{\infty} a_n$.

Ex. $a_n = \frac{1}{n^{3/2}+2}$ and $b_n = \frac{1}{n^{3/2}}$.

Then $a_n < b_n$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

converges $\left(p = \frac{3}{2}\right)$, so does

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+2}.$$

Does $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ converge?

NO! $\underset{a_n}{\overset{\shortparallel}{\frac{1}{\sqrt{n}-1}}} > \underset{b_n}{\overset{\shortparallel}{\frac{1}{\sqrt{n}}}}$.

Note that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

(by p-test, $p = \frac{1}{2}$)

$\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ also diverges.

(We could also use Limit Comparison Test)

Limit Comparison Test.

Suppose $a_n, b_n > 0$ and

that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

Then either both

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge

or both diverge.

Ex. Does $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+2}$ converge?

a_n

Set $b_n = \dfrac{\sqrt{n^2}}{n^2} = \dfrac{n}{n^2} = \dfrac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\dfrac{\sqrt{n^2+3}}{n^2+2}}{\dfrac{1}{n}} = \frac{n\sqrt{n^2+3}}{n^2+2}$$

$\to 1$ as $n \to \infty$. Hence the

Limit Comparison Test

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n^2+3}}{n^2+2} \text{ diverges}$$

since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Absolute Convergence

We say $\sum_{n=1}^{\infty} a_n$ converges absolutely

if $\sum_{n=1}^{\infty} |a_n|$ converges.

Thm. If $\sum_{n=1}^{\infty} a_n$ converges absolutely,

then $\sum_{n=1}^{\infty} a_n$ converges.

Ex. Show $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

and $\left| \dfrac{\sin n}{n^2} \right| \le \dfrac{1}{n^2}$.

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

so does $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ (Comparison Test)

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

which implies that

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ also converges.

Alternating Series.

If $\{b_n\}$ is decreasing

and $\lim_{n\to\infty} b_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Set $S = \sum_{n=1}^{\infty} (-1)^n b_n$. Then

$$|R_N| = \left| \sum_{n=1}^{N} (-1)^n b_n - S \right| < b_{N+1}.$$

Ex. If $S = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^3}$, find the smallest N

so that $|R_N| < .01$

$b_3 = \frac{1}{27}$

$b_4 = \frac{1}{64}$

$b_5 = \frac{1}{125} < .01 = \frac{1}{100}$

$\therefore \underline{\underline{|R_4| < \frac{1}{100}}}$ $N = 4$

Ex. Show $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges

Use the Roo Test

$$a_n = \left(\frac{e}{n}\right)^n$$

$$(a_n)^{\frac{1}{n}} = \frac{e}{n}$$

$$\therefore \lim_{n \to \infty} a_n^{\frac{1}{n}} = \lim_{n \to \infty} \frac{e}{n} = 0$$

Hence $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$ converges.

Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$,

suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

2. If $L > 1$, or if $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Power Series

Let $f(x) = \sum_{n=1}^{\infty} \frac{(-2)^{n+1}(x-2)^n}{\sqrt{n}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+2}|x-2|^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{2^{n+1}|x-2|^n}$$

$$= 2\sqrt{\frac{n}{n+1}}\,|x-2| \rightarrow 2|x-2|$$

as $n \rightarrow \infty$

$\therefore$ Power Series converges if

$2|x-2| < 1$, i.e., if $|x-2| < \frac{1}{2}$

$R = \frac{1}{2}$ or if $\frac{3}{2} < x < \frac{5}{2}$

If $x = \frac{5}{2}$, then series is

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}\left(\frac{5}{2}-2\right)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n\cancel{+}1}(-2)}{\sqrt{n}},$$

which converges by the

Alt. Series Test

If $x = \frac{3}{2}$, then series is

$$\left(\text{since} \left(\frac{3}{2}-2\right)^n = \left(-\frac{1}{2}\right)^n\right)$$

$$\sum_{n=1}^{\infty} \frac{(-2)^{n+1}\left(-\frac{1}{2}\right)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-2)}{\sqrt{n}},$$ which diverges

by p-test, $p = \frac{1}{2}$.

Ex. Express $\int_0^1 \frac{\sin x - x}{x^3}\,dx$

as a Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x - x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x - x}{x^3} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n+1)!}$$

$$\int \frac{\sin x - x}{x^3}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)(2n+1)!}$$

$$\therefore \int_0^1 \frac{\sin x - x}{x^3}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!\,(2n-1)}$$

Ex. Compute the Maclaurin series of $(1-2x)^{-\frac{1}{2}}$

$$(1+u)^{-\frac{1}{2}} = 1 + \left(\frac{-1}{2}\right)u + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)^2 u^2}{2!}$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)u^3}{3!} + \cdots$$

Set $u = -2x$:

$$(1-2x)^{-\frac{1}{2}} = 1 + x + \frac{1\cdot 3}{2!}x^2 + \frac{1\cdot 3\cdot 5}{3!}x^3 \cdots$$

$$+ C$$

Ex. Find the Maclaurin series

$\frac{1}{1-u} = 1+u+u^2+\ldots$

of $\frac{1}{4-3x} = \frac{1}{4}\cdot\frac{1}{1-\frac{3x}{4}}$

$$= \frac{1}{4}\left(1 + \frac{3x}{4} + \left(\frac{3x}{4}\right)^2 + \left(\frac{3x}{4}\right)^3 + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{3^n x^n}{4^{n+1}} + C$$

What is the radius of convergence?

Converges if $\left|\frac{3x}{4}\right| < 1$

OR $|x| < \frac{4}{3}$ $\therefore R = \frac{4}{3}$

Also, remember a series converges conditionally if it converges, but does NOT converge absolutely.

EXAMPLE $\sum (-1)^n \cdot \frac{1}{\sqrt{n}}$