MA 261

Multivariate Calculus

David Catlin

Office: 744 MATH Bldg.

email: catlind@purdue.edu

Office hours

Tues	11:30-12:30
Wed	2:30-3:30
Thurs	11:30-12:30

Class Web Page

math.purdue.edu/~catlin

Click on MA261 Web Page
under Spring Semester 2018

The class web page has a link to the official course web site. The official web site lists course ground rules and other course information.

We use WebAssign for homework

HW from Friday and Monday lectures is due Tuesday by 11:00 pm

HW from Wednesday lecture is due Thursday by 11:00 pm

No makeups for HW or Quizzes

3 lowest HW scores and 3 lowest QUIZ scores will be dropped.

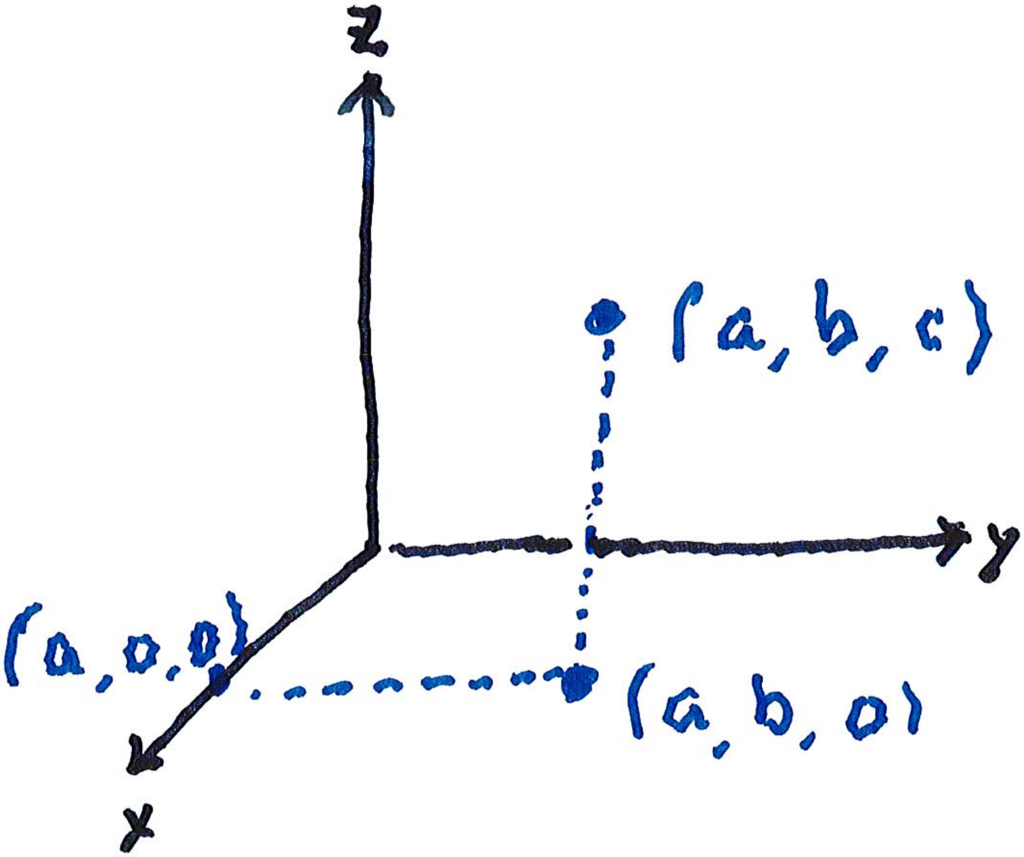
Please read Ground Rules at the official course web site and look at the course calendar.

All lectures will be posted at our class web site

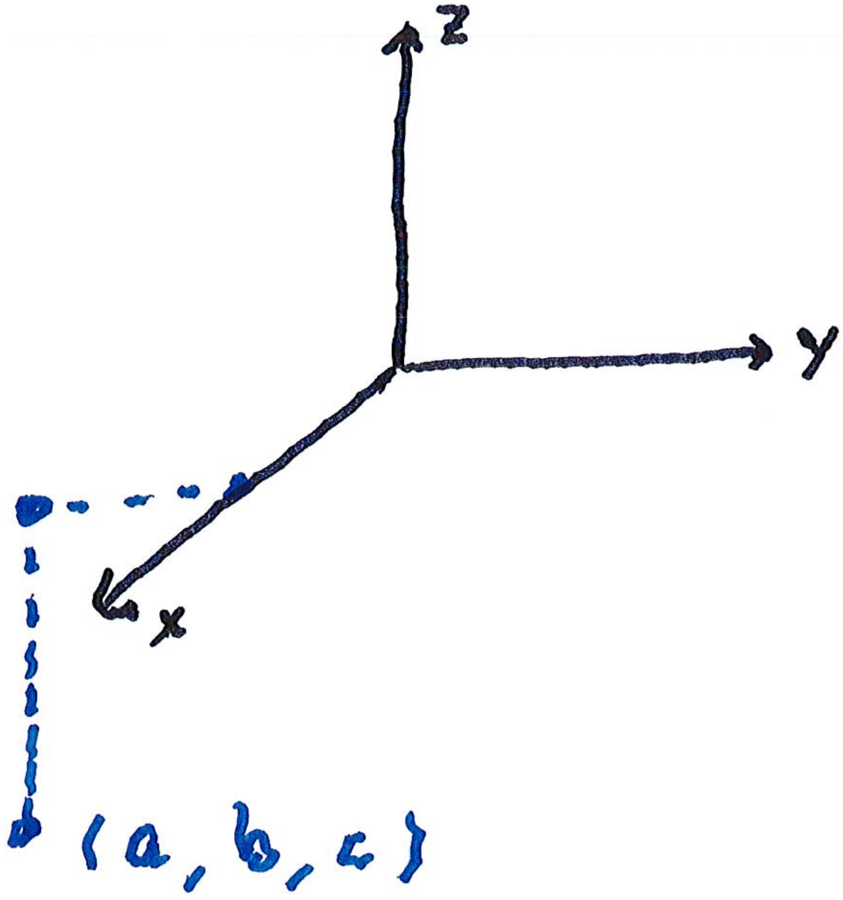
12.1 — 12.4 Review

12.1 3-dimensional coordinates

Each point in 3-dim. space can be written as $P(a, b, c)$



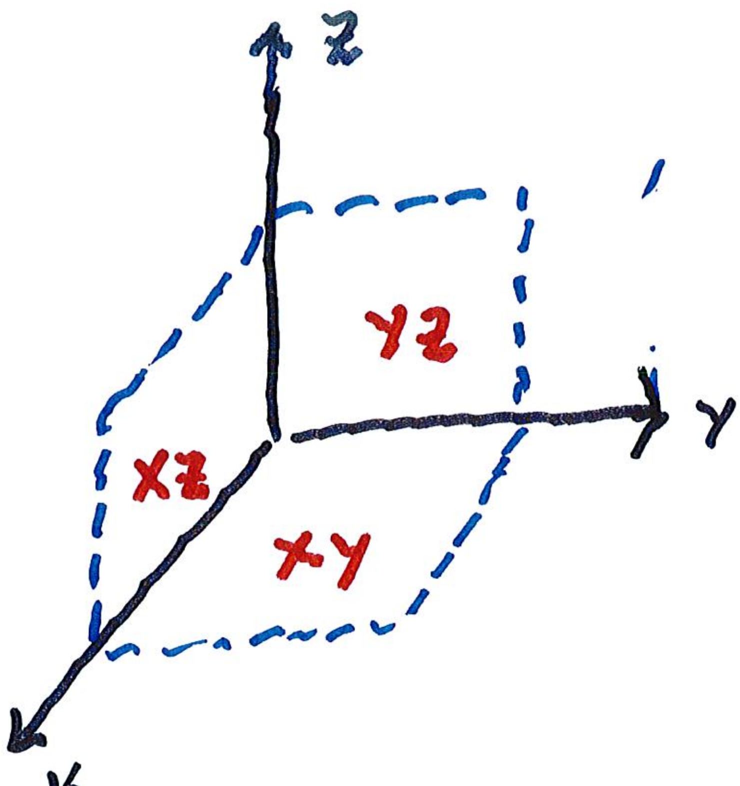
Suppose $a > 0$ and $b, c < 0$



Note that the xy-plane is $z = 0$

The xz-plane is $y = 0$

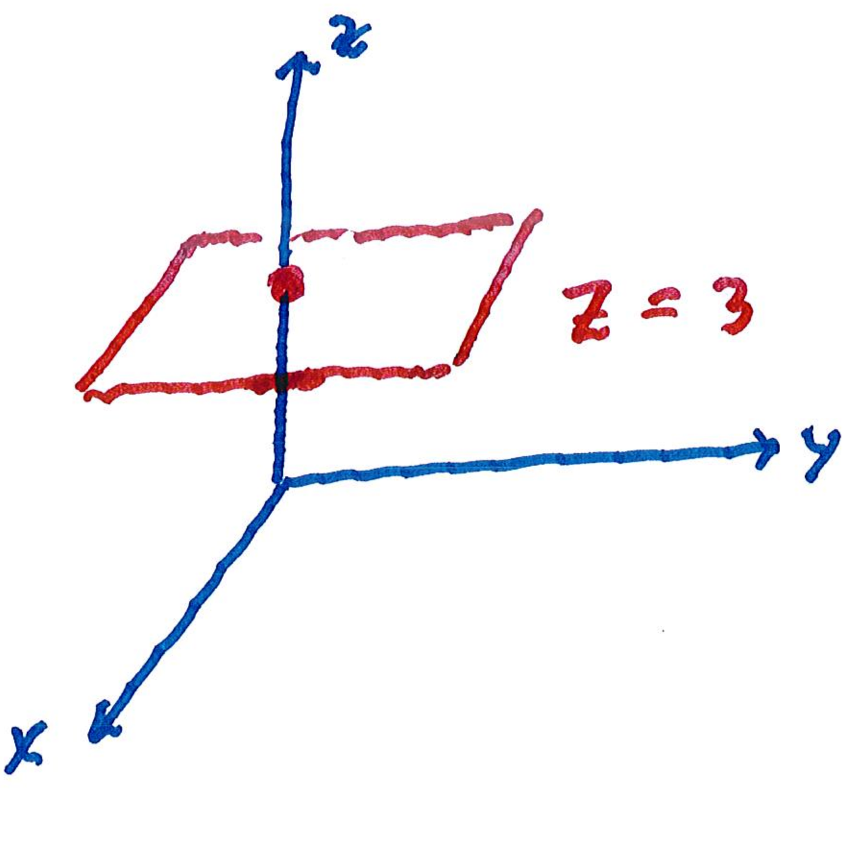
The yz-plane is $x = 0$



The equation $z = 3$

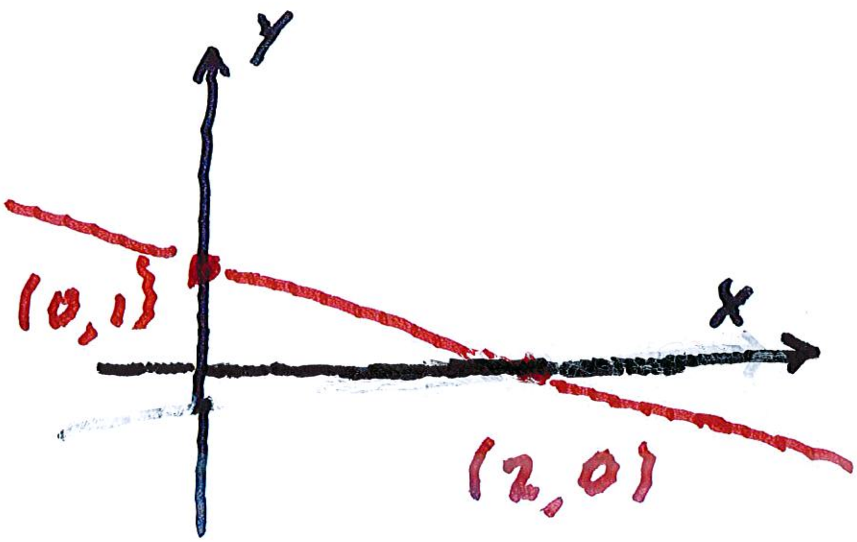
defines a flat plane

through $(0, 0. 3)$

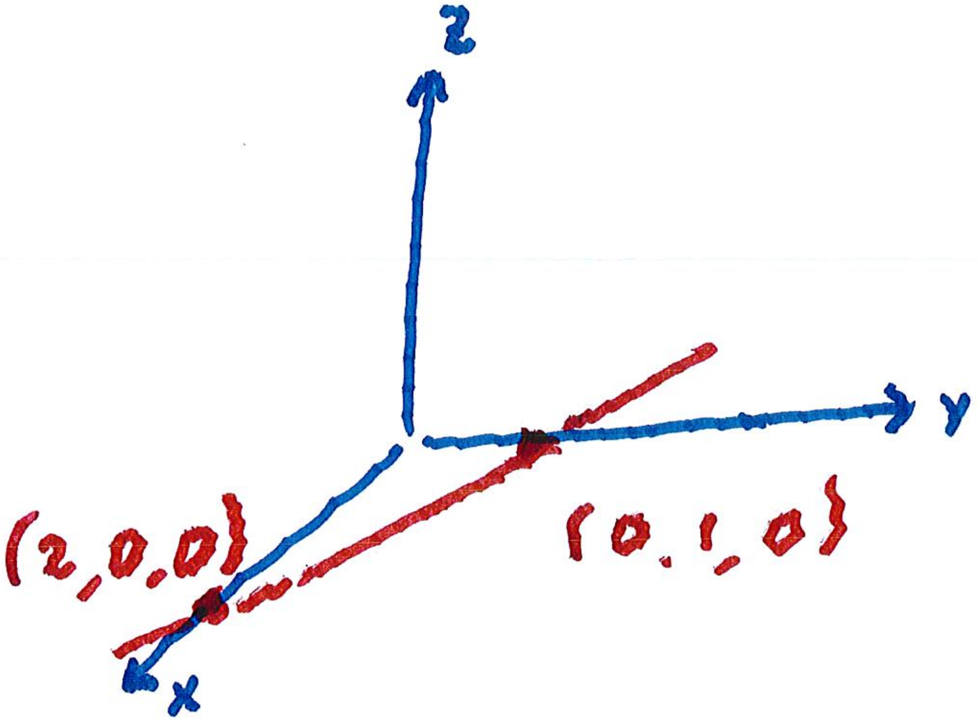


Ex. Sketch $x + 2y = 2$

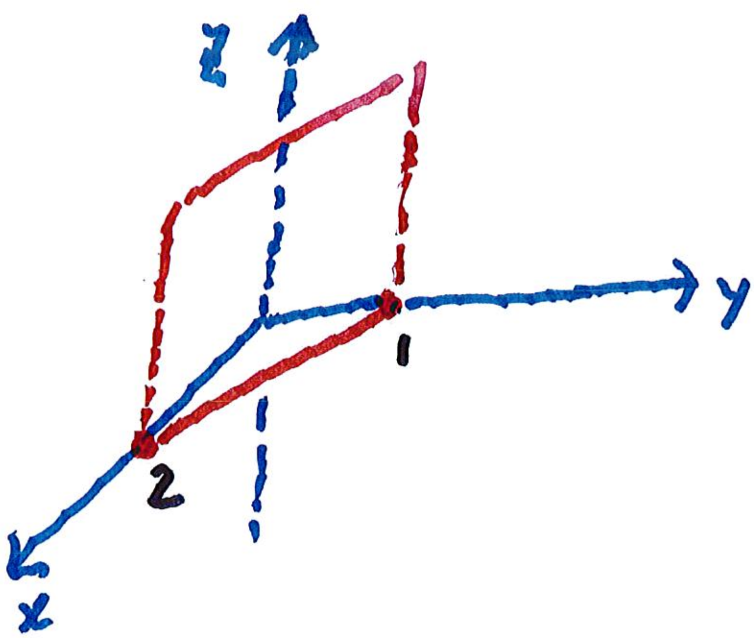
(ind. of z)



Put this line in xy-plane:

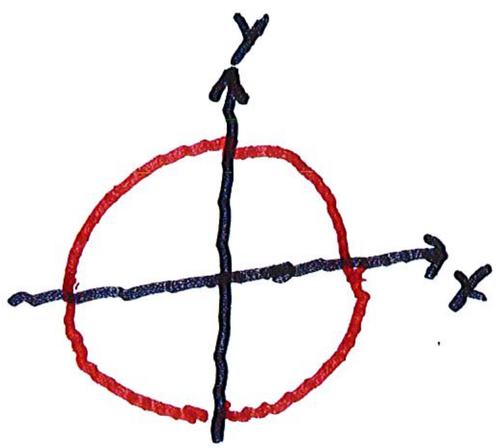


Slide the line up and down



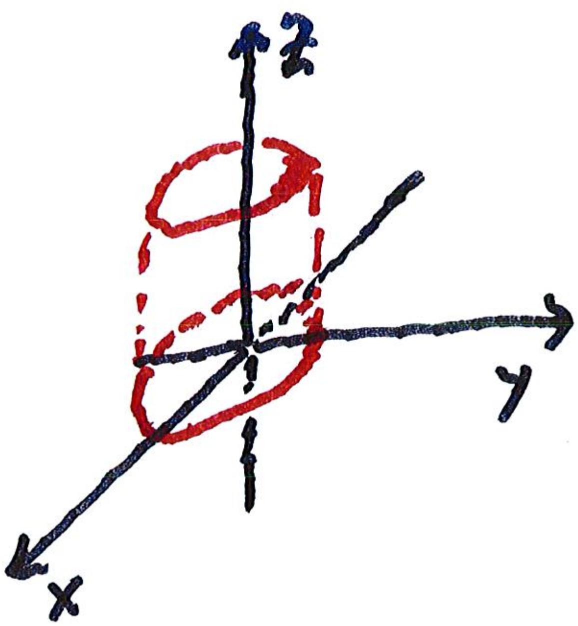
Ex. Sketch $x^2 + y^2 = 4$

Same idea:



$x^2 + y^2 = 4$

(radius = 2)



(ind. of z)

The distance between

$P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Ex. Find the radius and center of

$$x^2 + y^2 + z^2 + 2x - 4y + 4z = 3$$

$$(x+1)^2 + (y-2)^2 + (z+2)^2$$

$$= 1 + 4 + 4 + 3 = 12$$

$\therefore$ radius $= \sqrt{12}$

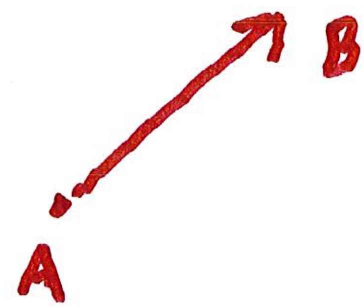
center $= (-1, 2, -2)$

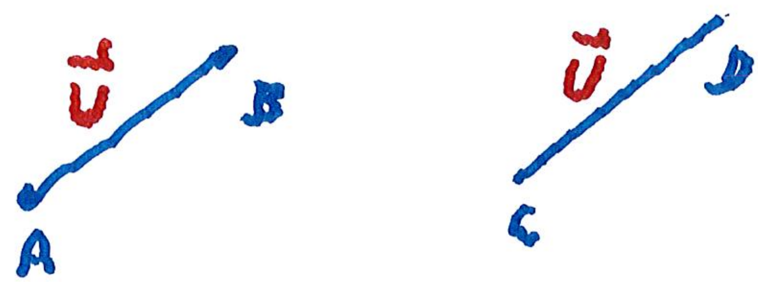
12.2 Vectors

Given points A and B,

we can form a displacement

vector $\overrightarrow{AB}$



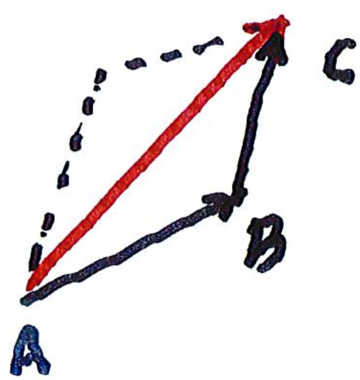


If $\overrightarrow{AB}$ is a translate of $\overrightarrow{CD}$, we consider $\overrightarrow{AB}$ and $\overrightarrow{CD}$ to be the same vector.

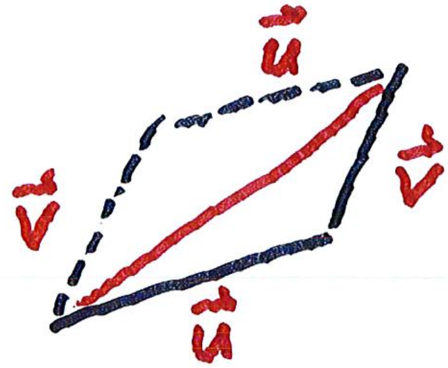
A is the initial point and B is the terminal point of $\overrightarrow{AB}$

Vector Addition is defined by the

Parallelogram Law:

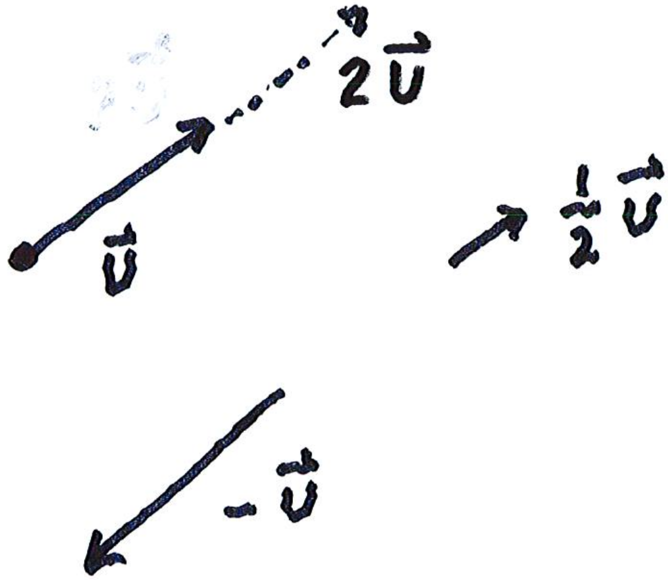


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$$\therefore \vec{u} + \vec{v} = \vec{v} + \vec{u}$$

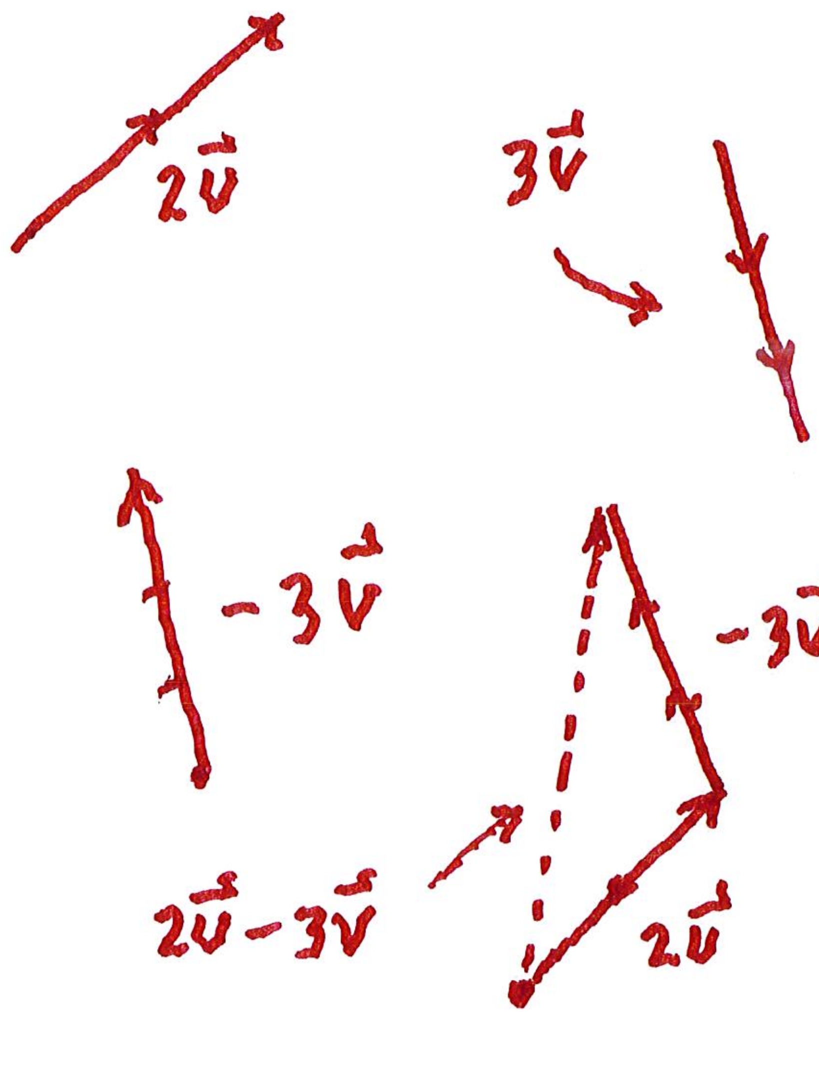
We can multiply $\vec{v}$ by any number c



If $\vec{u}$ is 

and $\vec{v}$ is  ,

what is $2\vec{u} - 3\vec{v}$?



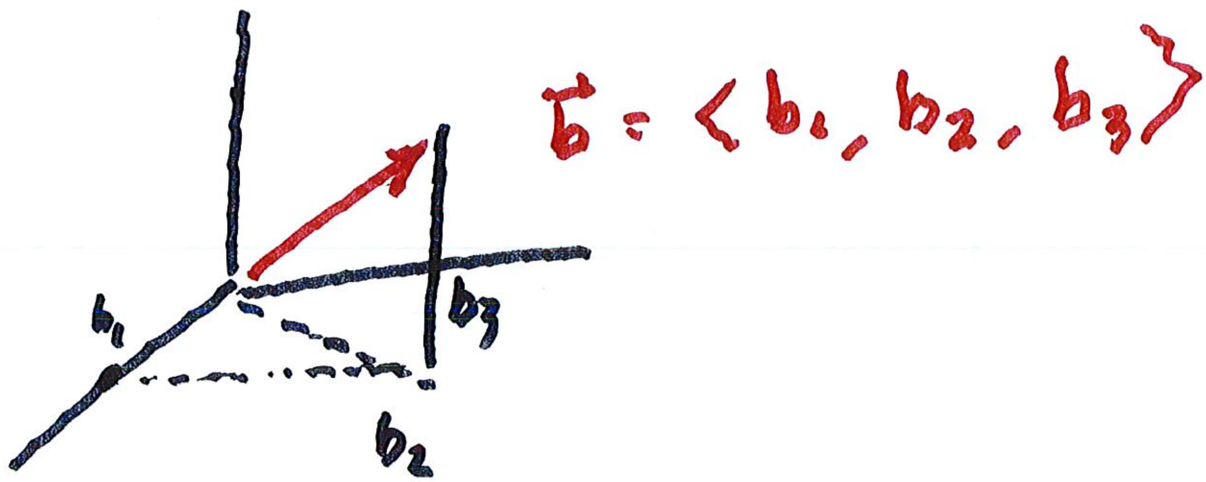
Components

For vectors in $\mathbb{R}^2$:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$ absolute value

If $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$



$\vec{a}$ and $\vec{b}$ are called

position vectors.

In $\mathbb{R}^2$, $\vec{a} + \vec{b}$

$= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$

$= \langle a_1 + b_1, a_2 + b_2 \rangle$

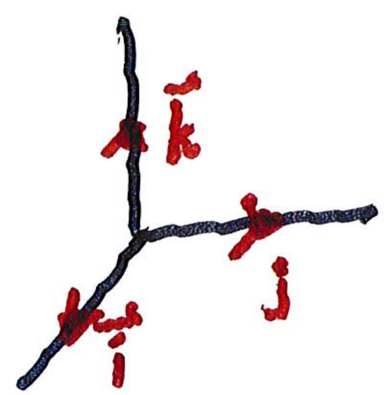
and $c\vec{a} = c\langle a_1, a_2\rangle$

$= \langle ca_1, ca_2\rangle$

Standard Basis Vectors

$\vec{i} = \langle 1, 0, 0\rangle \quad \vec{j} = \langle 0, 1, 0\rangle$

and $\vec{k} = \langle 0, 0, 1\rangle$



$\vec{V}$ is a unit vector if

$|\vec{v}| = 1.$

Ex. Find a vector $\vec{w}$ of length 3 that points in the opposite direction from

$\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$$

$$\vec{u} = \frac{1}{\sqrt{14}} \left(2\vec{i} + \vec{j} - 3\vec{k} \right)$$

is a unit vector

$$\vec{v} = \frac{3}{\sqrt{14}} \left(2\vec{i} + \vec{j} - 3\vec{k} \right)$$

has length 3.

$$\vec{w} = \frac{-3}{\sqrt{14}} \left(2\vec{i} + \vec{j} - 3\vec{k} \right)$$

12.3 Dot Product

Suppose

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ and

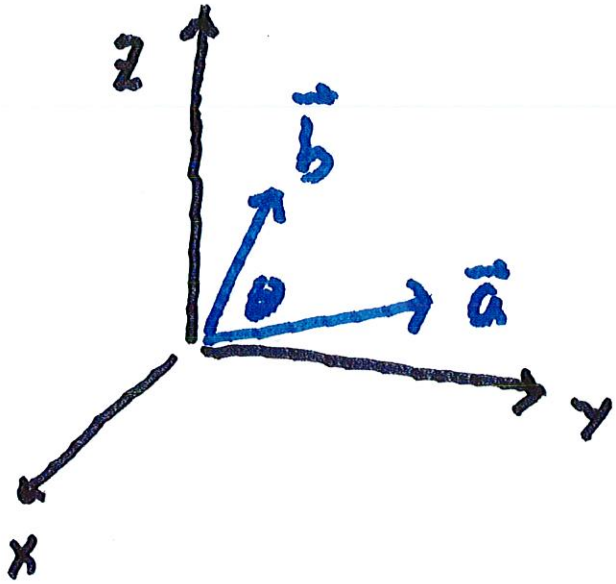
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of

$\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Geometric Meaning



Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \qquad (1)$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$

then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

and (1) is still true.

Ex. Find the angle θ between

$\vec{a} = \langle 3, -1 \rangle$ and $\vec{b} = \langle 2, 2 \rangle$

$\vec{a} \cdot \vec{b} = 3 \cdot 2 + (-1) \cdot 2 = 4$

$$|\vec{a}| = \sqrt{9+1} = \sqrt{10}$$

$$|\vec{b}| = \sqrt{4+4} = \sqrt{8}$$

$$\therefore \quad 4 = \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= \sqrt{10}\sqrt{8}\cos\theta$$

$$\rightarrow \quad 4 = \sqrt{80}\cos\theta$$

$$4 = 4\sqrt{5}\cos\theta$$

$$\therefore \cos\theta = \frac{1}{\sqrt{5}} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

Suppose $\vec{a}$ and $\vec{b}$ satisfy

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = 0$$

$$\therefore \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular

Conversely, if $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

Ex. Show $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$

and $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

are perpendicular.

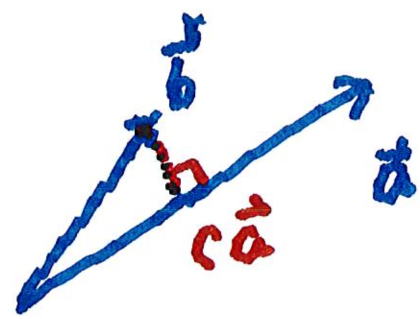
$\vec{a} \cdot \vec{b} = 2 \cdot 1 - 3 \cdot 1 + 1 \cdot 1 = 0$

$\Rightarrow \theta = \dfrac{\pi}{2}$

Scalar Projections

Find c so that

$\vec{b} - c\vec{a}$ is $\perp$ to $\vec{a}$



$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} = c$$

$$\left(\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2 \right)$$

$$\therefore \operatorname{proj}_{\vec{a}} \vec{b} = c\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

This is the vector projection

of $\vec{b}$ onto $\vec{a}$

We can also write

$$c\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|}$$

$\dfrac{\vec{a}}{|\vec{a}|}$ is a unit vector

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ is the scalar projectio

It gives the size of $\text{proj}_{\vec{a}}\vec{b}$

in the direction of $\vec{a}$

Ex. Find $\text{comp}_{\vec{a}}\vec{b}$ if

$\vec{a} = 2\vec{i} + \vec{k}$

and $\vec{b} = 3\vec{i} + \vec{j} + 2\vec{k}$

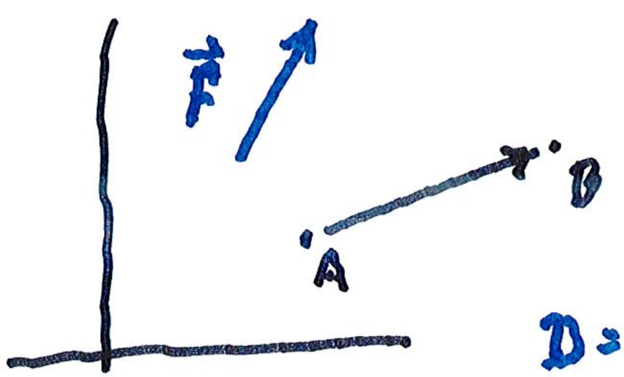
$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{6+2}{\sqrt{4+1}} = \frac{8}{\sqrt{5}}$$

$$\text{comp}_{\vec{a}}\vec{b}$$

$$\text{proj}_{\vec{a}}\vec{b} = \frac{8}{5}\left(2\vec{i} + \vec{k}\right)$$

Work

Suppose that there is a constant force $\vec{F}$ and that a particle moves from A to B



$D = \overline{AB}$ = displacement vector.

The work done by the force is

$$W = \vec{F} \cdot \vec{D}$$

12.4 Cross Product

Suppose $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are

vectors in $\mathbb{R}^3$

We define

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2 , a_3 b_1 - a_1 b_3 , a_1 b_2 - a_2 b_1 \rangle$$

cross product

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

or $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Ex. Compute $\langle 2, 1, -2 \rangle \times \langle 1, 3 -1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (-1+6)\vec{i} - (-2+2)\vec{j} + (6-1)\vec{k}$$

$$= 5\vec{i} + 5\vec{k}$$

Fact: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

and $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$$(\vec{a} \times b) \cdot \vec{a}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= (a_2 b_3 - a_3 b_2) a_1 - (a_1 b_3 - a_3 b_1) a_2$$

$$+ (a_1 b_2 - a_2 b_1) a_3 = 0 .$$

$\therefore$ $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

Similarly

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

Ex. Find a vector $\vec{v}$ that is $\perp$ to

$\langle 1, 2, 3 \rangle$ and $\langle 2, 2, -1 \rangle$.

$\vec{a}$ $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

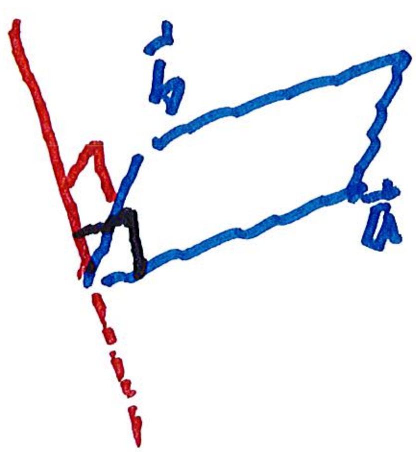
$$= \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= -8\vec{i} - (-1-6)\vec{j} + (2-4)\vec{k}$$

$$= \underline{-8\vec{i} + 7\vec{j} - 2\vec{k}} = \vec{v}$$

$\therefore$ $\vec{a} \times \vec{b}$ lies on line

that is $\perp$ to $\vec{a}$ and $\vec{b}$



Fact: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

where θ = angle between

$\vec{a}$ and $\vec{b}$.

Also, $|\vec{a} \times \vec{b}|$ = area of

parallelogram spanned

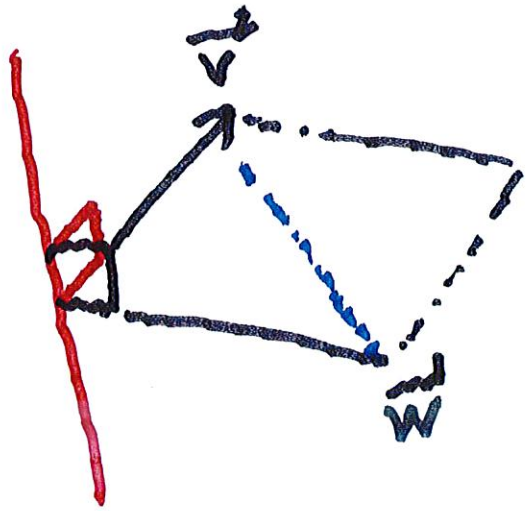
by $\vec{a}$ and $\vec{b}$

Ex. Find the area of the triangle with vertices at $P(2,1,-1)$ $Q(1,1,2)$ and $R(3,2,1)$.

$\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$

$\vec{v} = \langle -1, 0, 3 \rangle$ and

$\vec{w} = \langle 1, 1, 2 \rangle$



$\vec{a} \times \vec{b}$ is $\perp$ to plane spanned by $\vec{v}$ and $\vec{w}$

Area of Δ spanned by $\vec{v}$ and $\vec{w}$ $= \frac{1}{2}$(Area of Parallelogram)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -3\vec{i} - (-5)\vec{j} + (-1)\vec{k}$$

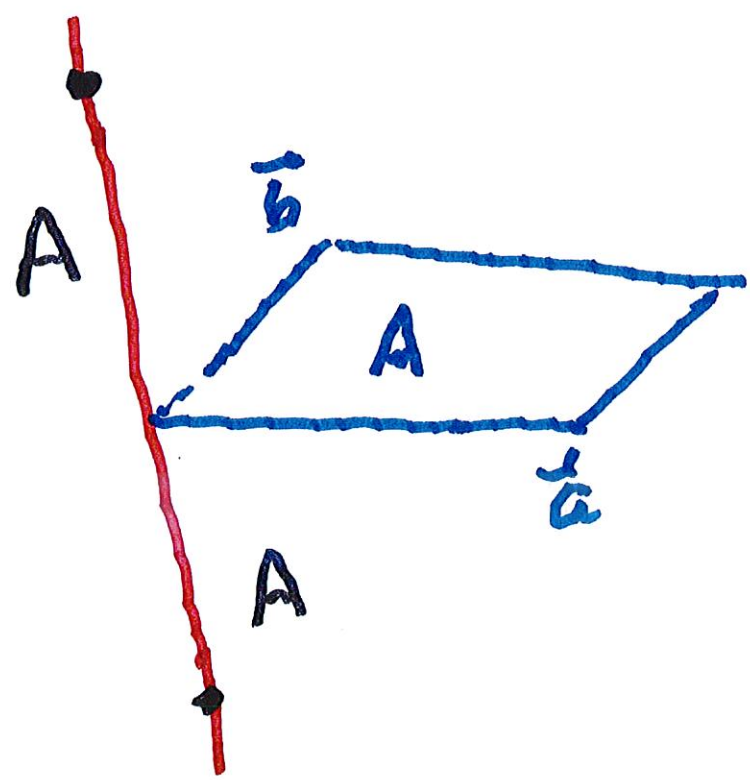
$$= -3\vec{i} + 5\vec{j} - \vec{k}$$

$$|\vec{v} \times \vec{w}| = \sqrt{9 + 25 + 1}$$

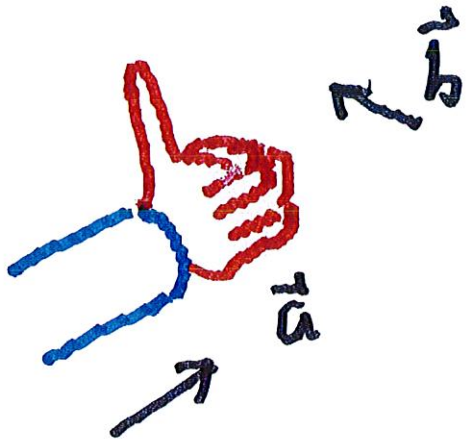
Area of $\Delta = \frac{1}{2}\sqrt{35}$

Now we know $\vec{a} \times \vec{b}$

lies at one of 2 locations



Right Hand Rule



Fingers curl from $\vec{a}$ to $\vec{b}$

$\Rightarrow \vec{a} \times \vec{b}$ points in

direction of thumb

Triple Product

Given vectors

$\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} \cdot (\vec{b} \times \vec{c})$ (the triple product)

equals volume of parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$