14.2 Limits and Continuity

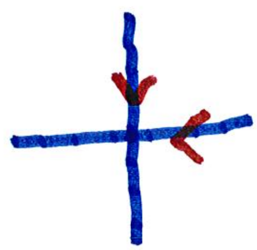
Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

What is $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Does the limit exist ?

Along the x-axis

$f(x,0) = 0$

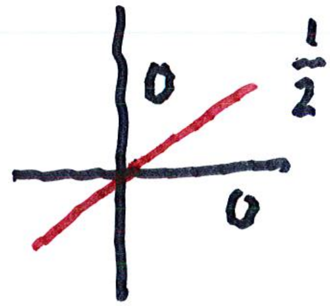


Along the y-axis,

$$f(0,y) = \frac{0\cdot y}{y^2} = 0$$

So, a good guess would be

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$$



Let's try

$y = x$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

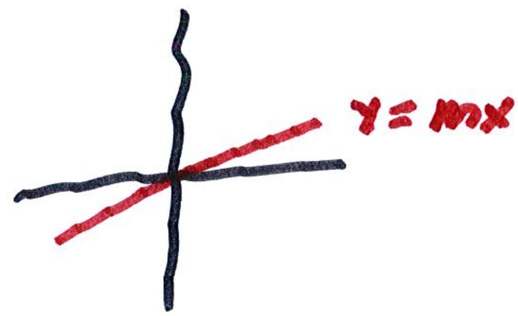
D. N. E.

Idea:

The $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

should be the same for

all lines $y = mx$



Ex. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - 2y^2}{x^2 + y^2}$

On x-axis $\dfrac{x^2 - 0}{x^2} = 1$

On y-axis $\dfrac{0 - 2y^2}{0 + y^2} = -2$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2} \quad \text{D.N.E.}$$

What is the correct definition?

Def'n. Suppose $f(x,y)$ is defined for all (x,y) near, (but not for (a,b)) We say that

Ex. Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Does $\lim f(x,y)$ exist as $(x,y) \to (0,0)$

Try $y = mx$.

$$f(x, mx) = \frac{x(mx)^2}{x^2 + (mx)^4} =$$

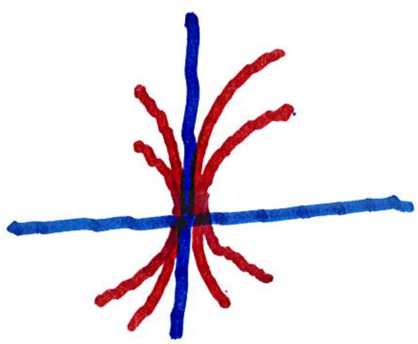
$$= \frac{m^2 x^3}{x^2 + m^4 x^4} = \frac{m^2 x}{1 + m^4 x^2}.$$

Should we say $f(x,y) \to 0$ on each line through $(0,0)$

But suppose $x = my^2$.

$$f(my^2, y) = \frac{my^2 \cdot y^2}{m^2y^4 + y^4}$$

$$= \frac{m}{m^2+1}$$



Note that $f(x,y)$ is a constant on each curve $x = my^2$.

In general, if $f(x,y) \longrightarrow L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

If the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b).

Consider $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$

$$\left| \frac{3x^2y}{x^2+y^2} \right| \leq \frac{3(x^2+y^2)\cdot\sqrt{x^2+y^2}}{x^2+y^2}$$

Note that

$|y| \leq \sqrt{x^2+y^2}$

$$\le \frac{3\,(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2} = 3\,(x^2+y^2)^{\frac{1}{2}} \to 0$$

as $(x,y) \to (0,0)$

Using the above intuitive definition, one can show

(i) $\lim\limits_{(x,y)\to(a,b)} x = a$

(ii) $\lim\limits_{(x,y)\to(a,b)} y = b$

(iii) $\lim\limits_{(x,y)\to(a,b)} c = c$

Continuity

We say a function of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

Using the usual properties of limits, one can see that sums, differences and products and quotients of continuous functions are also continuous,

{provided the denominator is $\neq 0$.}

A polynomial is a sum of terms of the form $cx^m y^n$

For ex, $f(x,y) = x^3 - 2xy + y^2 - 4$ is a polynomial, and therefore continuous everywhere. A rational function is a quotient of polynomials

$$g(x,y) = \frac{xy^2 - y^3}{x^4 + y^2 - 3}$$

Ex. Where is the function not defined?

$$\frac{2xy^2}{3x^2+y^4}$$ (only at $(0,0)$)

It's continuous at each point (a,b) if $(a,b) \neq (0,0)$.

But $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ <u>is</u> continuous at $(0,0)$ if we set $f(0,0) = 0$.

A function $f(x, y, z)$ has a limit at (a, b, c) if $f(x, y, z)$ approaches the same number L, no matter how $(x, y, z) \to (a, b, c)$.

We say f is continuous at (a, b, c) if $\lim_{(x,y,z) \to (a, b, c)} f(x, y, z)$

$$= f(a, b, c)$$

11. $$\lim \frac{y^2 \sin^2 x}{x^4+y^4}$$

$$\sin x \sim x \to \sin^2 x \approx x^2$$

$\therefore$ Same as

$$\lim \frac{y^2 x^2}{x^4+y^4}$$

D.N.E.

13 $$\lim \frac{xy}{\sqrt{x^2+y^2}}$$

$$\cancel{|2xy| \le \sqrt{\ }} \quad |xy| \le x^2+y^2$$

$$\frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$$

$$\therefore \lim_{(x,y)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$