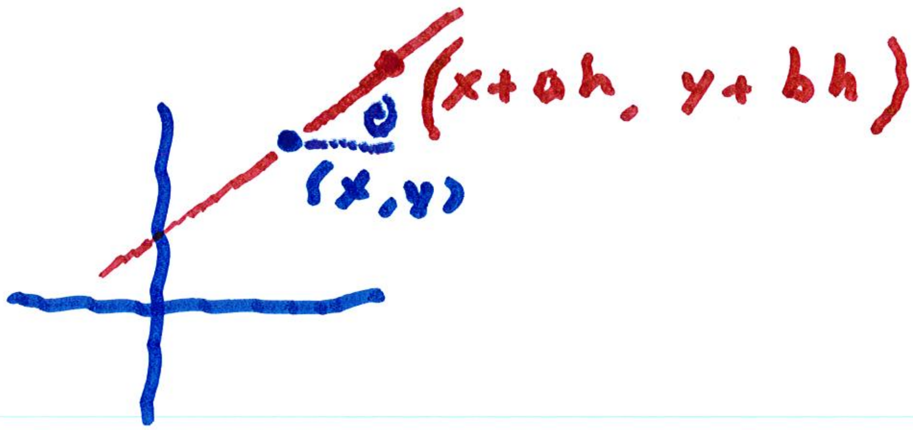
14.6 cont'd.

If $\vec{u} = \langle a, b \rangle$ is a unit vector, and if $f(x,y)$ is a function, then

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$D_{\vec{u}} f$ = rate of change of $f(x+ha, y+hb)$ at $h=0$



Ex. Find the directional derivative $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^3 - 2xy + 4y$ and $\vec{u}$ = unit vector given by angle $\theta = \frac{-\pi}{6}$. What is $D_{\vec{u}} f(2,1)$?

$$\nabla f = \langle 3x^2 - 2y \, , \, -2x + 4 \rangle$$

at $(2,1)$ in the direction

$\vec{u}$ that makes an angle of

$-\frac{\pi}{6}$ $\quad \nabla f = \langle 10 \, , \, 0 \rangle \cdot \left\langle \frac{\sqrt{3}}{2} \, , \, -\frac{1}{2} \right\rangle$

$$= 5\sqrt{3}$$

Recall that

$$D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

$$\left\{ \begin{array}{l} \text{since} \\ \vec{u} = \langle a, b \rangle \end{array} \right\}$$

If we write

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

This is called the gradient of f at (x,y)

$$\therefore D_{\vec{v}} f(x,y) = \nabla f_{(x,y)} \cdot \vec{u}$$

Ex. Use the gradient to compute the dir. deriv. of $f(x,y) = 2x^2y - y^2$ at $(-2, 1)$ in the direction of $\langle -1, 1\rangle$.

We let $\vec{u} = \dfrac{-\vec{i} + \vec{j}}{\sqrt{2}}$ which is a unit vector.

When $x = -2$ and $y = 1$,

$$\frac{\partial f}{\partial x} = 4xy = -8$$

$$\frac{\partial f}{\partial y} = 2x^2 - 2y = 6$$

Hence $\nabla f = -8\vec{i} + 6\vec{j}$

$$\therefore D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$= (-8\vec{i} + 6\vec{j}) \cdot \left(\frac{-i}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)$$

$$= \quad = \frac{8+6}{\sqrt{2}} = \frac{14}{\sqrt{2}}.$$

The case when $f = f(x, y, z)$

is similar. If we write

$\vec{u} = \langle a, b, c \rangle$ (a unit vector)

then

$$D_{\vec{u}} f(x, y, z)$$

$$= \lim_{h \to 0} \frac{f(x + ha, y + hb, z + hc) - f(x, y, z)}{h}$$

Using vector notation,

with $\vec{x}_0 = \langle x, y, z \rangle$ and

$\vec{u} = \langle a, b, c \rangle$, we get

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

Similarly, we can define

gradient of f by

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Ex. Find the dir. - der of

$f(x, y, z) = xy + yz + xz$

at $(1, -1, 3)$ in the direction of

$(2, 4, 5)$.

$$\nabla f(x, y, z) = \langle y+z, x+z, x+y \rangle$$

$$= \langle 2, 4, 0 \rangle$$

Note that $\vec{u} = \left\langle \frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, \frac{5}{\sqrt{45}} \right\rangle$

$$\therefore D_{\vec{u}} f = \frac{4}{\sqrt{45}} + \frac{16}{\sqrt{45}} = \frac{20}{\sqrt{45}}$$

Thm. Suppose f is a differentiable function of 2 or 3 variables. The maximal value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta$$

$$= |\nabla f| \cos\theta$$

Max. Val. of $\cos\theta$ is when $\theta = 0$. $\theta = 0 \leftrightarrow$ which means $\vec{u} = \dfrac{\nabla f}{|\nabla f|}$

Ex. Find the unit vector that gives the largest value of $D_{\vec{u}} f$, when $f(x,y,z) = xe^{yz}$

$$f_x = e^{yz}, \quad f_y = xze^{yz}, \quad f_z = xye^{yz}$$

$$\nabla f = e^{yz}\langle 1, xz, xy\rangle$$

Dir. of maximal increase is

$$\frac{\langle 1, xz, xy\rangle}{\sqrt{1 + x^2z^2 + x^2y^2}}$$

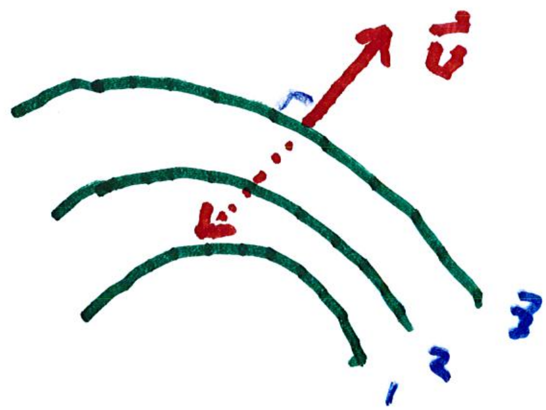
The direction of maximal decrease occurs when $\cos\theta = -1$, where θ is the angle between $\vec{u} = \dfrac{\nabla f}{|\nabla f|}$

Thus $\vec{u} = -\dfrac{\nabla f}{|\nabla f|}$

$\nabla f \cdot \vec{u}$

$= \dfrac{\nabla f \cdot \nabla f}{|\nabla f|}$

$= \dfrac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$



Suppose S is a surface with equation $F(x, y, z) = k$. Let C be any curve which lies on S and which is defined by $\vec{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle$

Hence $F(x(t), y(t), z(t)) = k$.

If we differentiate in t,

we get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

Since $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t)\rangle$

and $\nabla F = \langle F_x, F_y, F_z\rangle$,

we get

$$\nabla F \cdot \vec{r}\,'(t) = 0$$

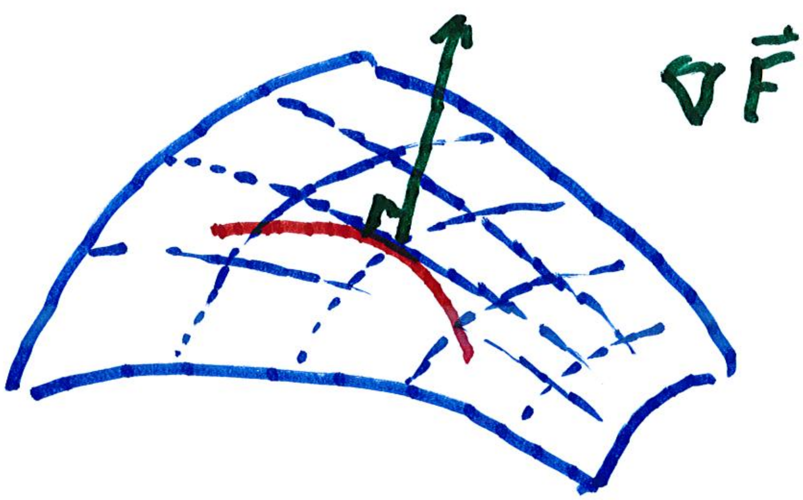
This shows us that $\nabla F(x_0, y_0, z_0)$

is $\perp$ to the tangent vector

vector $\vec{r}'(t_0)$ to any curve

C on S that passes

through (x_0, y_0, z_0)



$\therefore$ ∇F is the vector that

is $\perp$ to the plane through

(x_0, y_0, z_0),

we get

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0)$$

$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to S at P

is $\perp$ to the tangent plane,

and satisfies

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Now consider the special case when a surface is defined as the graph of a function, i.e. $\underbrace{z - f(x,y) = 0}_{F(x,y,z)=0}$

Then

$$F_x(x,y,z) = -\frac{\partial f}{\partial x}(x,y)$$

$$F_y(x,y,z) = -\frac{\partial f}{\partial y}(x,y)$$

$$F_z(x,y,z) = 1$$

The tangent plane can be described by

$$-f_x(x_0, y_0)(x - x_0)$$

$$-f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0.$$

or

$$z - z_0 = f_x(x_0, y_0)(x - x_0)$$

$$= f_y(x_0, y_0)(y - y_0)$$

Ex. Suppose an ellipsoid S

is described by

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Therefore

$F_x(x,y,z) = \frac{x}{2}$ $F_y(x,y,z) = 2y$ and

$$F_z(x,y,z) = \frac{2z}{9}$$

$\therefore F_x(-2,1,-3) = -1$ $F_y(-2,1,-3) = 2$

$$F_z(-2,1,-3) = -\frac{2}{3}$$

It follows that the tangent plane at $(-2, 1, 3)$ is

$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$