

14.7 Maximum and Minimum Values.

Def'n. A function $f(x, y)$

has a local maximum at

(a, b) if $f(x, y) \leq f(a, b)$

when (x, y) is near (a, b) .

The number $f(a, b)$ is

called the local maximum value.

Similarly, if $f(a, b) \leq f(x, y)$
for (x, y) near (a, b) , then
 f has a local minimum,
and $f(a, b)$ is a local minimum
value.

Theorem. If f has a local
minimum or local maximum
at (a, b) and the first-
order partial derivatives

exist at (a, b) , then

$$\underline{f_x(a, b) = 0} \quad \text{and} \quad \underline{f_y(a, b) = 0}$$

Ex. If we set $g(x) = f(x, b)$,

and if f has a local

maximum or minimum at

(a, b) , then g has a local

maximum (or minimum) at a ,

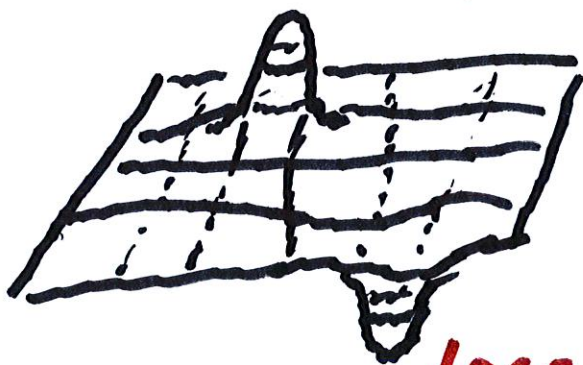
so $f_x(a, b) = 0$. Also, if we

set $G(y) = f(a, y)$, then

$G'(b) = 0$, which implies

that $f_y(a, b) = 0$.

local maximum



local minimum

A point (a, b) is a critical

point of f if

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

either

(1) One of the partial

derivatives $f_x(a, b)$

or $f_y(a, b)$ does not

exist, OR

(2) if both of the partial

derivatives exist, then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

At a critical point, then
 f can have a local maximum
or local minimum or
neither.

Ex. Let $f(x, y) = x^2 - 4x + 6y$
 $+ y^2 + 2,$

Find all crit. pts.

then $f_x = 2x - 4$

$$f_y = 6 + 2y$$

If $f_x = 0$, then $x = 2$, and

if $f_y = 0$, then $y = -3$

If we complete the square,

$$f(x, y) = (x-2)^2 + (y+3)^2 - 11.$$

$\therefore f$ has a local minimum
at $(2, -3)$. In fact

f has an absolute

minimum at $(2, -3)$.

because $f(x, y) \geq f(2, -3) = -11$

for all (x, y) in \mathbb{R} .

Ex. Find all extreme values
of $f(x, y) = x^2 - y^2$.

$$f_x(x, y) = 2x$$

$$f_y(x, y) = -2y.$$

\therefore Only critical point
is $(0, 0)$.

$$f(x, 0) = x^2, \text{ so } f(x, 0)$$

takes on all positive values.

Also, $f(0, y) = -y^2$ takes

on all negative values.

$\therefore f$ has no extreme values.

Note $z = x^2 - y^2$ has

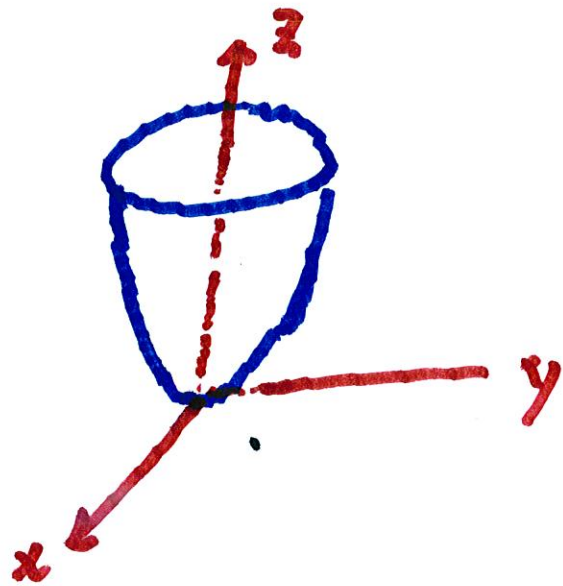
a saddle at $(0, 0)$



On the other hand,

$z = x^2 + y^2$ is a paraboloid

and has a global minimum
at $(0, 0)$



In 1-variable calculus,

$y = f(x)$ has a local minimum
at x_0
if $f'(x_0) = 0$.

For a function $f(x, y)$ we have:

Second Derivative Test.

Suppose the second partial derivatives of f are continuous on a disk about (a, b) , and suppose that (a, b) is a critical point of f , then we define

The Second Derivative Test.¹⁰

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(a) If $D > 0$ and $f_{xx} > 0$,

then $f(a, b)$ is a local minimum

(b) If $D > 0$ and $f_{xx}(a, b) < 0$,

then f has a local maximum

(c) If $D < 0$, then $f(a, b)$

is not a local maximum or

minimum

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Note 1 If $D < 0$, then one can show that the graph of f has a saddle point

Note 2 If $D = 0$ and f_{xx}

then the test gives no

information, i.e. f could

have a local min. or max

or (a, b) could be a saddle point.

Ex. Find all local maximum
and local minimum values
and saddle points:

$$\text{Ex } f(x, y) = xy(1-x-y)$$

$$= xy - x^2y - xy^2$$

$$f_x = y - 2xy - y^2 = y(1-2x-y)$$

$$f_y = x - x^2 - 2xy = x(1-2y-x)$$

Note $f_x = 0$ if $y = 0$ or $2x + y = 1$

and $f_y = 0$ if $x = 0$ or $x + 2y = 1$

There are 4 possibilities

1. $x=0, y=0$ $(0,0)$

2. $x=0, 1-2y=0$, i.e., $y=\frac{1}{2}$

3. $y=0, 1-2x=0$ $(\frac{1}{2}, 0)$

4. $1-2x-y=0$ } $(\frac{1}{2}, 0)$
 and $1-2y-x=0$ } $\rightarrow x=\frac{1}{3}, y=\frac{1}{3}$

$(\frac{1}{3}, \frac{1}{3})$

Now we calculate D

$$f_x = -2xy \quad \therefore f_{xx} = -2y$$

$$f_{xy} = 1 - 2x - 2y$$

$$f_{yy} = -2x$$

$$\text{Hence } D = \begin{vmatrix} -2y & 1 - 2x - 2y \\ 1 - 2x - 2y & -2x \end{vmatrix}$$

$$= (1 - 2x - 2y)^2$$

$$= 0 \cdot 1^2 = 0$$

$$\text{Ex. } f(x, y) = 3x - x^3 - 2y^2 + y^4$$

1. Find critical pts

$$f_x: 3 - 3x^2 = 0 \rightarrow x = \pm 1$$

$$f_y = -4y + 4y^3 = 4y(-1 + y^2)$$

$$y = 0, 1, \text{ or } -1$$

$$\therefore (1, -1) \quad (-1, -1)$$

$$(1, 0) \quad (-1, 0)$$

$$(1, 1) \quad (-1, 1)$$

$$f_{xx} = -6x$$

$$f_{yy} = (-4 + 12y^2)$$

$$f_{xy} = 0$$

$$\therefore D = 24x - 72xy^2$$

1. At $(1, -1)$, $D = 24 - 72 = -48$
SADDLE PT

2. At $(1, 0)$ $D = 24$

and $f_{xx} = -6$
 < 0

\rightarrow local max at $(1, 0)$

3. At $(1, 1)$,

$\rightarrow D = -48$

\therefore saddle point

4. At $(-1, -1)$

$\rightarrow D = -48 \rightarrow$ saddle pt

$$f_{xx} = +2y = -2 < 0 \therefore \text{Local Max}$$

5. At $(-1, 0)$

$D = -24 \rightarrow$ saddle point

6. At $(-1, 1)$

$$D = 48 \quad f_{xx} = -6x = 6$$

\therefore local minimum

Ex. Set $f(x, y) = (x-y)(1-xy)$

$$f(x, y) = x - y - x^2y + xy^2$$

$$f_x = 1 - 2xy + y^2 = 0$$

$$f_y = -1 - x^2 + 2xy = 0$$

$$\text{Add. } y^2 - x^2 = 0$$

$$y = x \rightarrow x^2 - 2x^2 + 1 = 0$$

$$\rightarrow x = \pm 1 \quad (1, 1) \text{ or } (-1, 1)$$

$$\text{If } y = -x.$$

$$\rightarrow 1 + 2x^2 + x^2 = 0$$

$$\rightarrow (x+1)^2 = 0 \rightarrow x = -1$$

$$\therefore (x, y) = (-1, 1)$$

\therefore 3 points

$$(1, 1), (-1, -1), (-1, 1)$$

$$f_{xx} = -2y \quad f_{yy} = 2x$$

$$f_{xy} = -2x + 2y$$

$$\therefore D = -4xy + 4(y-x)^2$$

$$= 4x^2 - 12xy + 4y^2$$

1. $(1, 1)$ $D = -4 \rightarrow$ saddle point

$$f(1, 1) = 0$$

2. $(-1, -1)$ $D = 4 - 12 + 4 = -4$

saddle pt.

$$f(-1, -1) = 0$$

3. $(-1, 1)$ $D = 20$ $f_{xx} = -2$

\therefore local maximum.

Pts. at $(0, 0)$,

or $(0, 1)$

or $(1, 0)$

$$\left. \begin{array}{l} \text{or } x + 2y = 1 \\ 2x + y = 1 \end{array} \right\} x = \frac{1}{3}, y = \frac{1}{3}$$

$$\left(\frac{1}{3}, \frac{1}{3} \right).$$

Calculate D:

$$f_{xx} = -2y \quad f_{yy} = -2x$$

$$f_{xy} = 1 - 2x - 2y$$

$$D = 4xy - (1 - 2x - 2y)^2$$

At $(0, 0)$ $D = 0 - 1 = -1$

saddle pt

At $(1, 0)$ $D = 0 - (1 - 2)^2 = -1$

saddle pt

At $(0, 1)$ $D = 0 - 1 = -1$

saddle pt

At $(\frac{1}{3}, \frac{1}{3})$

$$D = \frac{4}{9} - \left(1 - \frac{4}{3}\right)^2 = \frac{4}{9} - \frac{1}{9}$$

$$= \frac{1}{3} > 0$$

$$f_{xx} = -2 \cdot \frac{1}{3} < 0$$

$\therefore f$ has a local maximum

at $(\frac{1}{3}, \frac{1}{3})$