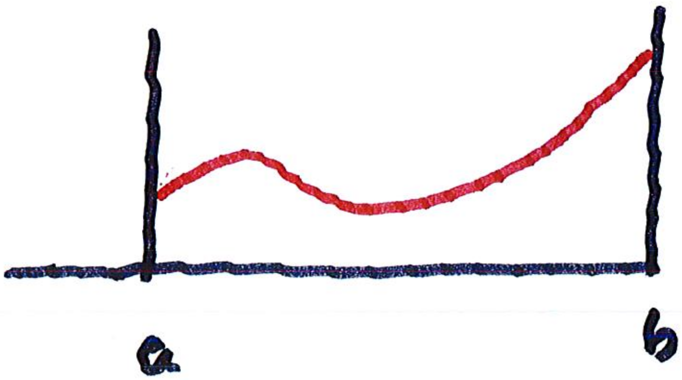
15.1 Double Integrals over Rectangles.

Remember, the integral in one variable is essentially the area under a curve {when $f(x) \geq 0$} between $x=a$ and $x=b$.



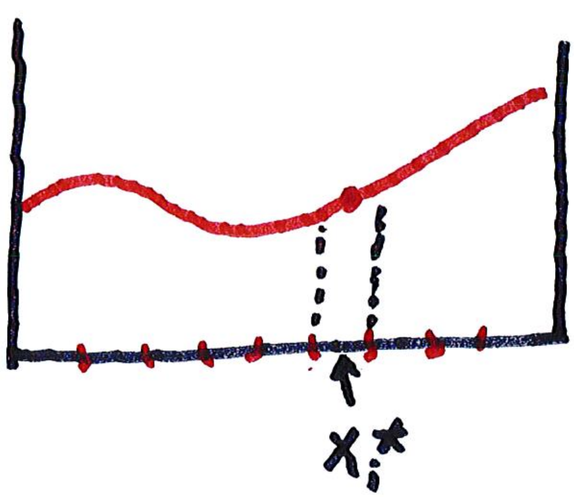
To compute the area, we

decompose $[a, b]$ into

n small intervals: $[x_{i-1}, x_i]$

for $1 \leq i \leq n$

and $x_i - x_{i-1} \leq \frac{b-a}{n}$



We let x_i^* be any random point

in $[x_{i-1}, x_i]$

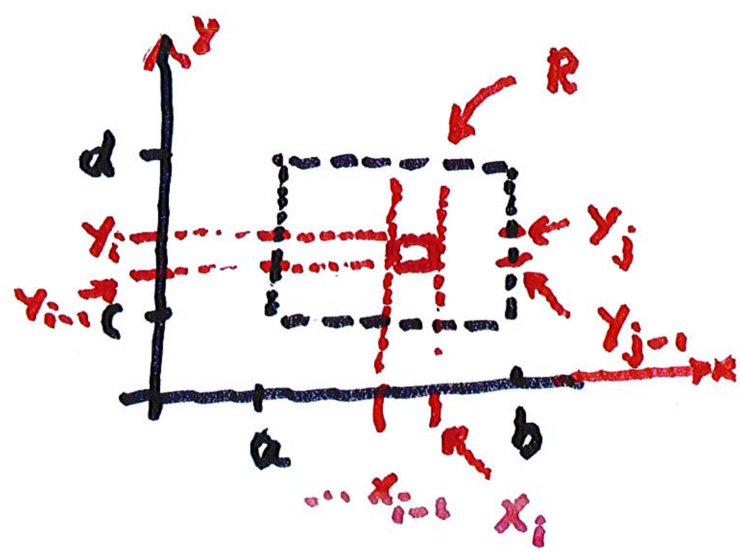
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{n=1}^{\infty} f(x_i^*)\,\Delta x,$$

where $\Delta = \frac{b-a}{n}$

Suppose that $f(x, y)$ is defined

for $\begin{cases} a \le x \le b & \text{and} \\ c \le y \le d \end{cases}$

Subdivide rectangle into smaller ones:

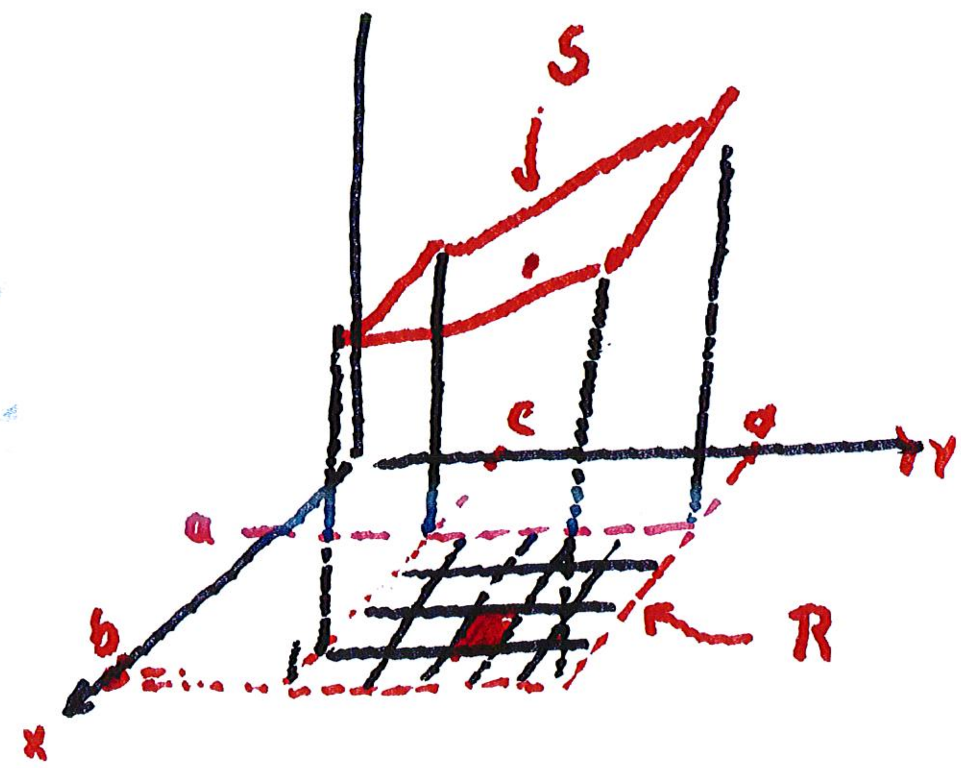


The small rectangle is defined by

$x_{i-1} \le x \le x_i$ and $y_{j-1} \le y \le y_j$

$\Delta x = \dfrac{b-a}{m}$ $\quad \Delta y = \dfrac{d-c}{n}$

The small rectangle is denoted by R_{ij}.



We want to compute the volume under S, $z = f(x,y)$

with $\left\{ \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix} \right\} = R$

Let (x_{ij}^*, y_{ij}^*) be a randomly selected point in R_{ij}

The volume above R_{ij} and below S is

$$\Delta V \approx f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$$

and the total volume is

approximately

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^{*}, y_{ij}^{*}) \Delta x \Delta y$$

If we let $m \to \infty$ and $n \to \infty$

then if $f(x,y)$ is continuous

at all (x,y) except for

" a small set ", then

the limiting value of the above double sum is

$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where $\Delta A = \Delta x \cdot \Delta y$

Note that the above definition of

$$\iint_R f(x,y)\,dA$$ makes sense

when $f(x,y)$

is ≥ 0 or ≤ 0.

Sometimes, instead of (x_{ij}^*, y_{ij}^*)

one uses (x_i, y_j) instead of

Upper Right Corner

(x_{ij}^*, y_{ij}^*)

or $(\bar{x}_i, \bar{y}_j)$, where

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} \quad \text{and}$$

$$\bar{y}_j = \frac{y_{j-1} + y_j}{2}.$$

This often gives a good approximation of $\iint_R f(x,y)\,dA$.

The average value of a function f on R is =

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA.$$

Area of R.

Properties of Double Integrals

(1) $$\iint_R [f(x,y) + g(x,y)]\, dA$$

$$= \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

(2) $$\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA.$$

(3) If $f(x,y) \geq g(x,y)$ in R, then

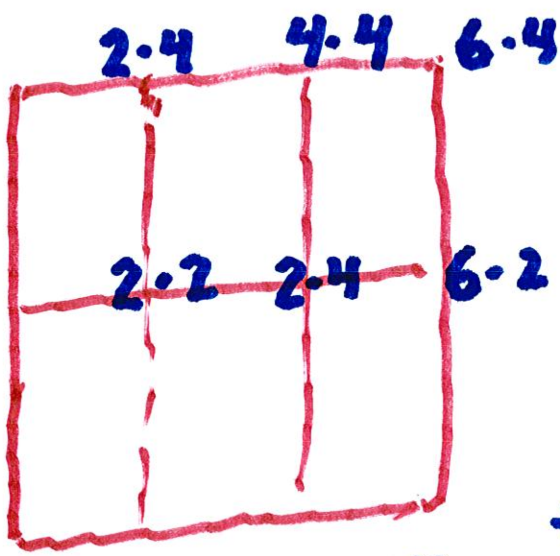
$$\iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.$$

Ex. Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle $R = \begin{cases} 0 \le x \le 6 \\ 0 \le y \le 4. \end{cases}$

Use Riemann Sum with

$m = 3$, $n = 2$.

Use upper right corner



Area = 4

$8 + 16 + 24$

$+ 4 + 8 + 12$

Estimate is

$$= (48 + 24) \cdot 4 = 288$$

For the Midpoint Rule:

$1 \cdot 3$	$3 \cdot 3$	$5 \cdot 3$
$1 \cdot 1$	$3 \cdot 1$	$5 \cdot 1$

Riemann Sum $4(1+3+5+3+\quad 9+\quad +15)$

$$= 144$$