Ex. Compute $\displaystyle\int_0^1 \int_0^1 \frac{x\,dy}{1+xy}\,dx$

Note that $\displaystyle\frac{\partial}{\partial y}\left(\ln(1+xy)\right)$

$$= \frac{x}{(1+xy)}$$

Hence,

$$\int_0^1 \int_0^1 \frac{\partial}{\partial y}\left(\ln(1+xy)\right) dy\,dx$$

$$= \int_0^1 \left(\ln(1+xy)\right)\Big|_{0=y}^{1=y}\,dx$$

$$= \int_0^1 \ln(1+x) - \ln 1\,dx$$

$$= \int_1^2 \ln t \quad dt$$

$$= \int_1^2 t \ln t - t \quad dt$$

$$= 2 \ln 2 - 2 + 1 = \underline{\underline{2 \ln 2 - 1}}$$

Ex. Compute $\displaystyle\int_0^1 \int_0^1 (s+t)^{1/2} \, ds \, dt$

$$= \int_0^1 \frac{2}{3}(s+t)^{3/2} \Big|_0^1 dt$$

$$\frac{2}{3}\int_0^1 (1+t)^{\frac{3}{2}} - t^{3/2} \quad dt$$

$$= \quad \frac{2}{3}\cdot\frac{2}{5}(1+t)^{5/2} - \frac{2}{3}\cdot\frac{2}{5}\, t^{5/2} \quad \Big|_0^1$$

$$= \frac{4}{15}\left(2^{5/2} - 1\right) - \frac{4}{15}\cdot 1$$

$$= \frac{4}{15}\left(2^{5/2} - 2\right)$$

Ex Compute $\int_0^1 \int_0^3 e^{x+3y}\, dy\, dx$

$$= \int_0^1 e^x \int_0^1 e^{3y}\, dy \quad dx$$

$$= \int_0^1 e^x \left(\frac{1}{3} e^{3y} \Big|_0^3 \right)$$

$$= \int_0^1 e^x \left(\frac{1}{3} e^9 - \frac{1}{3} \right) dx$$

$$= \left(\frac{e^9 - 1}{3} \right) \int_0^1 e^x\, dx$$

$$= \frac{e^9 - 1}{3} \cdot (e^1 - 1)$$

15.3 We learned how to integrate a function defined on a rectangle R

If $R = \left\{ (x,y) \,\middle|\, \begin{array}{l} a \le x \le b \\ c \le y \le d \end{array} \right\}$,

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

or

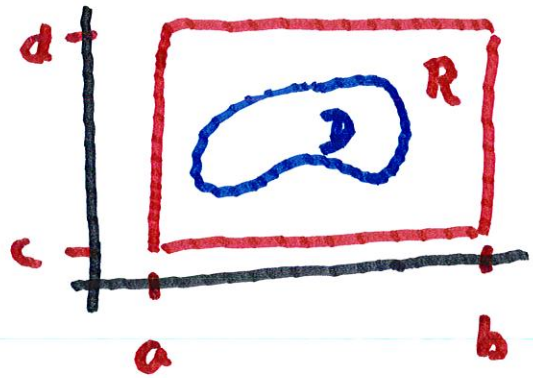
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

What if R is replaced by

a more complicated region?

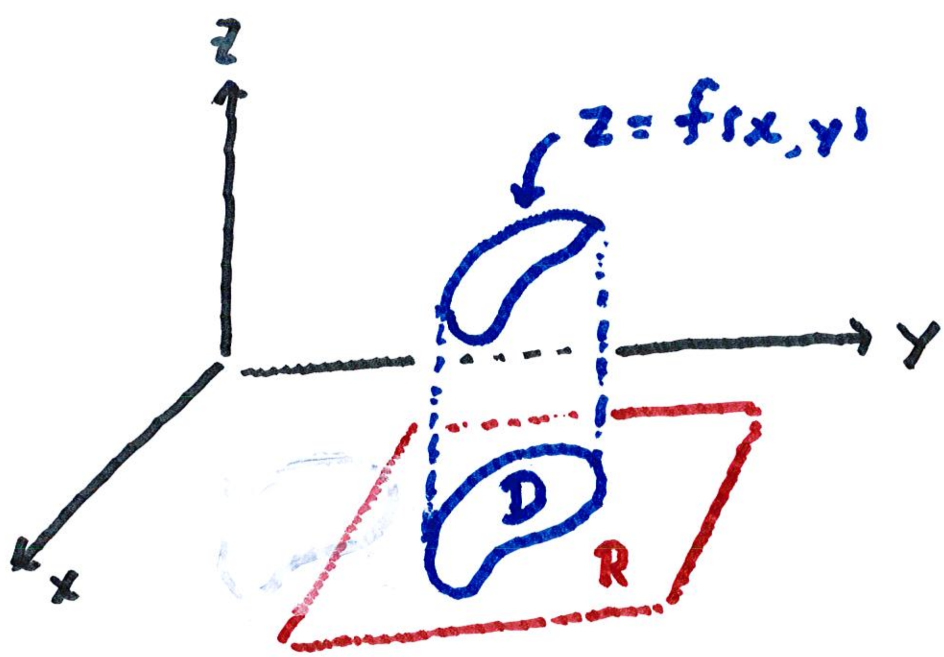
We can write

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \\ & \text{but not in } D \end{cases}$$



Geometrically

$$\iint f(x,y)\,dA = \text{volume of region above } D, \text{ under } z = f(x,y)$$

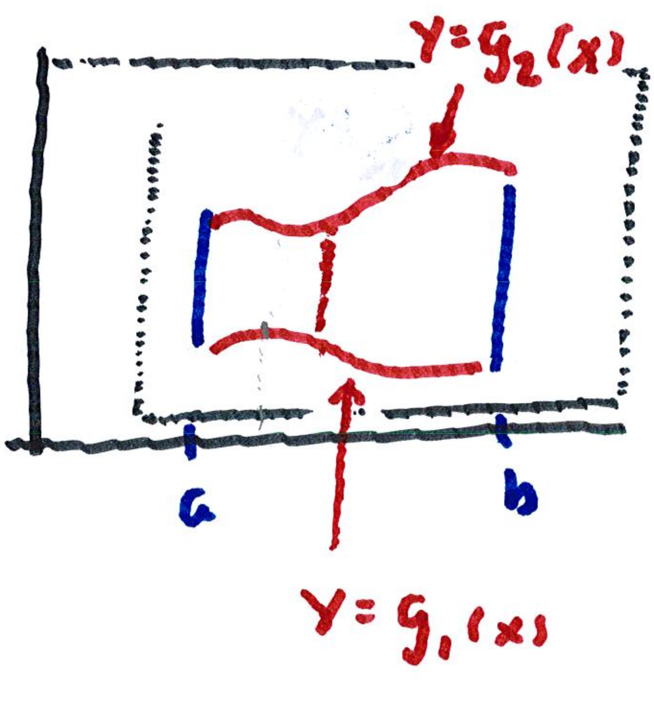


A plane region D is of type I

if it lies between the graphs

of two functions:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)$$



$$= \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{\text{y-integral}}\,dx$$

Ex Let $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ x^2 \le y \le x \end{array} \right\}$

Compute $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2 \, dy\, dx$

$$= \int_0^1 \left. \frac{xy^3}{3} \right|_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x \cdot x^3}{3} - \frac{x \cdot x^6}{3} \right) dx$$

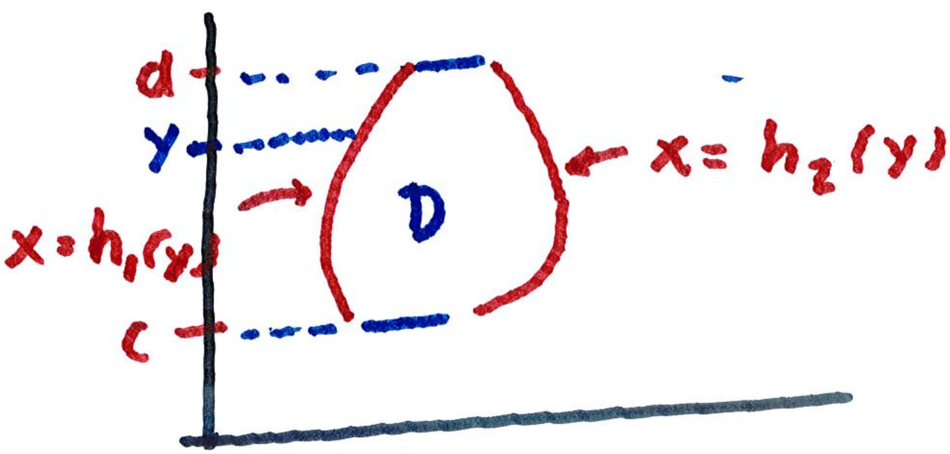
$$= \int_0^1 \frac{x^4}{3} - \frac{x^7}{3} \, dx$$

$$= \frac{x^5}{15} - \frac{x^8}{24} \Big|_0^1$$

$$= \frac{1}{15} - \frac{1}{24} = \frac{8-5}{120} = \underline{\underline{\frac{1}{40}}}$$

A region D is of type II

if $D = \left\{ (x,y) \,\middle|\, \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right.$



Ex. Let D be the region between

the graphs of $x = y^2$ and $x = 2y+3$

Compute $\displaystyle\iint_D 2x \, dA$

First, when do the curves

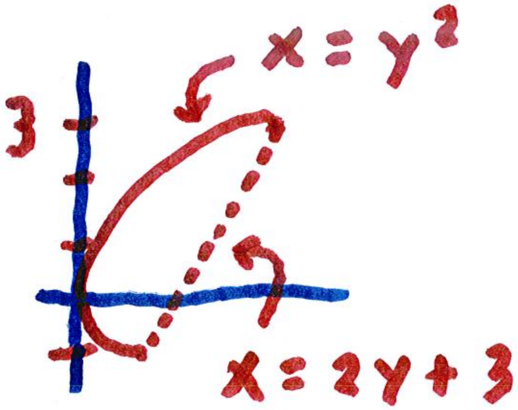
$x = y^2$ and $x = 2y+3$ coincide?

$$y^2 = x = 2y+3$$

$$\rightarrow \quad y^2 = 2y+3$$

$$y^2 - 2y + 3 = 0 \rightarrow (y-3)(y+1) = 0$$

$$\therefore \; y = -1, \; y = 3$$



Plug $y = 0$ into both equations.

$$0 = x \qquad 2 \cdot 0 + 3 = y$$

$\therefore$ Integral is

$$\int_{-1}^{3} \int_{y^2}^{2y+3} 2x \, dx \, dy$$

$$= \int_{-1}^{3} x^2 \Big|_{x=y^2}^{x=2y+3} dy$$

$$= \int_{-1}^{3} (2y+3)^2 - y^4 \, dy$$

$$= \int_{-1}^{3} -y^4 + 4y^2 + 12y + 9 \, dy$$

$$= -\frac{y^5}{5} + \frac{4y^3}{3} + 6y^2 + 9y \Big|_{-1}^{3}$$

9

$$= \left[-\frac{3^5}{5} + 36 + 54 + 27 \right]$$

$$- \left[\frac{1}{5} - \frac{4}{3} + 6 - 9 \right]$$

$$= \underline{\underline{-376}}$$

It is useful to view $\int_{-1}^{3} \int_{y^2}^{2y+3} 2x \; dx\, dy$

as a type II region

because the lower boundary has 2 parts.

Ex. Evaluate $\iint_D xy \, dA$.

where

D is bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$

$y^2 = x$ $\quad$ $x = 2y$

Usually it's better to avoid

square roots

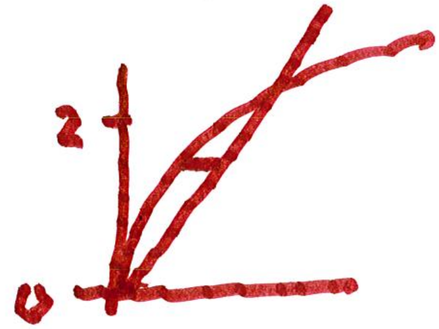
$$y^2 = x = 2y$$

$$\rightarrow y^2 - 2y = 0 \rightarrow \begin{cases} y = 0 \\ y = 2 \end{cases}$$

Plug $y = 1$ $\quad y^2 = 1 \quad 2 \cdot 1 = 2$

$\therefore$ $x = 2y$ is bigger (in x-direction

$$\int_0^2 \int_{y^2}^{2y} xy \, dx \, dy$$



$$= \int_0^2 \frac{x^2 y}{2} \Big|_{y^2}^{2y} \, dy$$

$$= \int_0^2 \frac{(2y)^2 y}{2} - \frac{(y^2)^2 y}{2} \, dy$$

$$= \int_0^2 2y^3 - \frac{y^5}{2} \, dy$$

$$= \frac{y^4}{2} - \frac{y^6}{12} \Big|_0^2 = \frac{8}{3}$$

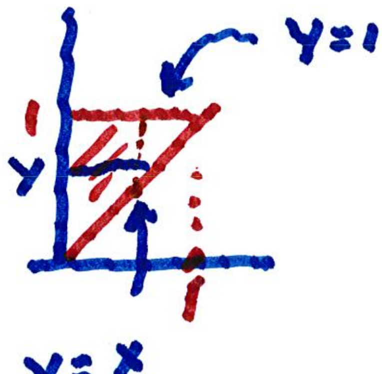
Ex. Sometimes only Type I or

Type II is possible

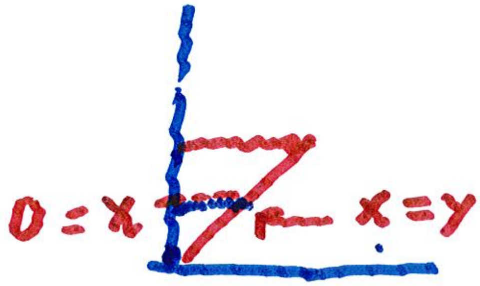
Evaluate $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$

$$= \iint_D \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \int_x^1 \sin y^2\, dy\, dx$$



$= ?$



$$\iint_{D} \sin(y^2)\, dA$$

$$= \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \Big[x \sin(y^2) \Big]_{x=0}^{x=y}$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$$= \frac{1}{2}\int_0^1 \sin(y^2)\, 2y\, dy$$

$$= -\frac{1}{2}\cos(y^2)\Big|_0^1$$

$$= -\frac{1}{2}\cos 1 + \frac{1}{2}\cos 0$$

$$= \frac{1}{2}(1-\cos 1)$$

Properties of Double Integrals

$$\iint_D [f(x,y) + g(x,y)]\, dA$$

$$= \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

And:

$$\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$$

If $f(x,y) \geq g(x,y)$, for (x,y) in D,

then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

Also if $D = D_1 \cup D_2$, where

D_1 and D_2 don't intersect, then (except on boundaries)

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

$$\iint_D 1 \, dA = \text{Area of } D$$

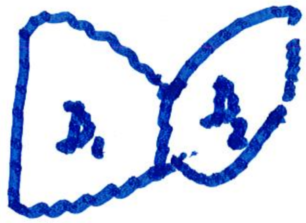
and if

$$m \leq f(x,y) \leq M. \quad \text{for all } (x,y) \text{ in } D.$$

then

$$m \,\text{Area}(D) \leq \iint_D f(x,y)\, dA$$

$$\leq M \,\text{Area}(D)$$



$D = D_1 \cup D_2$

31. Find volume of region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, and $z = 0$ in the first octant.

The region on the x-y plane

is $\left\{ (x,y) : \begin{array}{l} 0 < x < 1 \\ 0 < y < \sqrt{1-x^2} \end{array} \right\} = D^+$

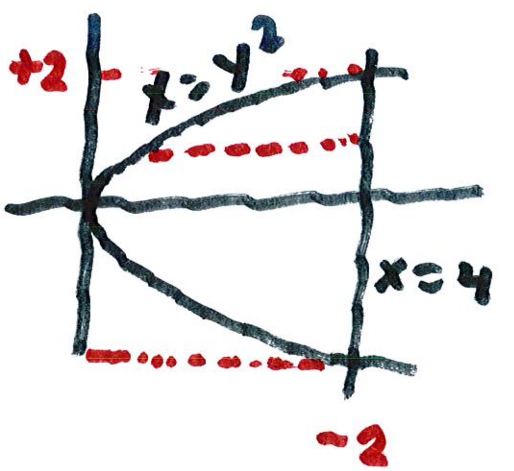
$$\text{Vol} = \int_0^1 \int_0^{\sqrt{1-x^2}} y \, dy \, dx$$

$$= \int_0^1 \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1-x^2}{2} \, dx = \frac{x}{2} - \frac{x^3}{6} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Find the volume of the region under the surface $z = 1 + x^2 + y^2$ and above the region enclosed by $x = 4$ and $x = y^2$



$y^2 = x = 4$

$y = \pm 2$