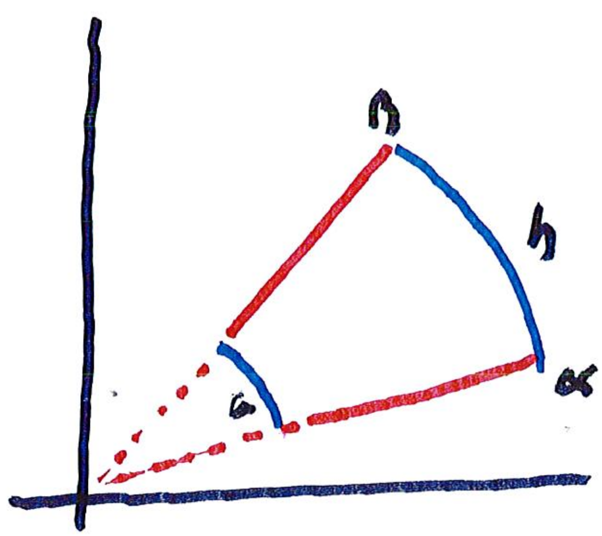
15.3

~~15.4~~ Double Integrals in Polar Coordinates.

Given a "polar rectangle".

how do we write it as a sum

of many small "polar rectangles



$\alpha < \theta < \beta$

$a \le r \le b$

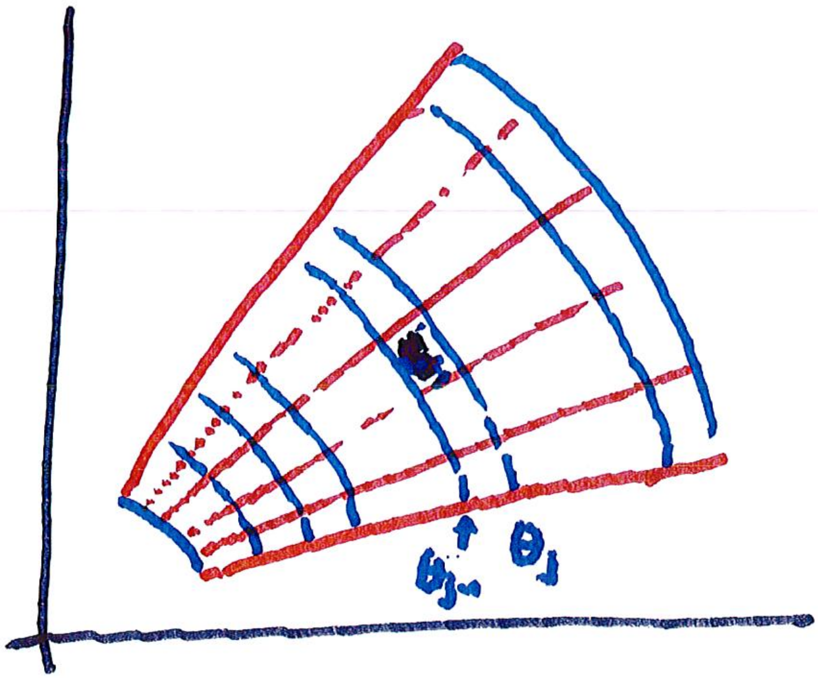
Subdivide

$$a = r_0 < r_1 < r_2 \ldots < r_{j-1} < r_j < \ldots < r_m = b$$

where $r_j - r_{j-1} = \Delta r = \dfrac{b-a}{m}$

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_{k-1} < \theta_k \ldots < \theta_n = \beta$$

where $\theta_k - \theta_{k-1} = \Delta\theta = \dfrac{\beta - \alpha}{n}$



$$\Delta A = \pi(r_j^2 - r_{j-1}^2) \cdot \frac{\Delta\theta}{2\pi}$$

$$= \pi(r_j + r_{j-1})(r_j - r_{j-1}) \cdot \frac{\Delta\theta}{2\pi}$$

If we assume that $r_{j-1} \approx r_j$ and that $\theta_k \approx 0$, then

$$\Delta A \approx r_j \, \Delta r \, \Delta \theta .$$

Now imagine that the height of the box above the small polar rectangle is

$$f(x, y) = f(r_j \cos \theta_k , r_j \sin \theta_k)$$

then the total volume of solid region is

$$V \approx \sum_{j=1}^{m} \sum_{k=1}^{n} f(r_j \cos\theta_k, r_j \sin\theta_k) \, r_j \Delta\theta \Delta r$$

Letting $m, n \to \infty$, then

$$V = \iint_R f(r\cos\theta, r\sin\theta) r \, dA$$

$dA = r \, dr \, d\theta$

Note that $r \, d\theta \, dr = dA$.

To complete the integral one includes the "fudge" factor r as above.

The main result about polar polar nates is:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$$

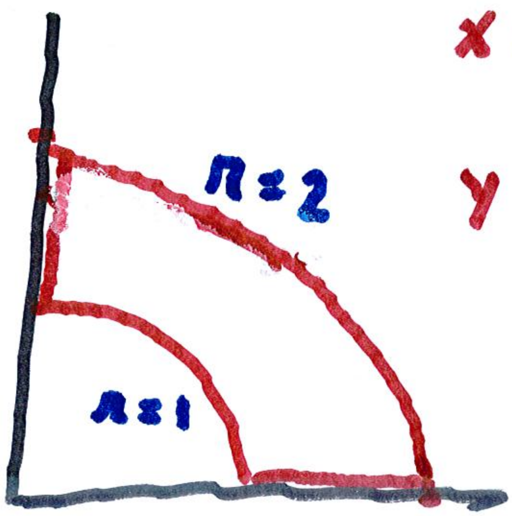
Note that

$dA = r\,dr\,d\theta$

We Compute the integral as follows:

$\iint_R (x^2+y)\,dA$, where R is the region in the first quadrant bounded by $x^2+y^2=1$ and $x^2+y^2=4$

Note that the above polar region is



$x = r\cos\theta$

$y = r\sin\theta$

We compute that

$$I = \int_0^{\frac{\pi}{2}} \int_1^2 r^2\cos^2\theta + r\sin\theta$$

$$r\,dr\,d\theta$$

$$I_1 = \int_0^{\frac{\pi}{2}} \int_1^2 r^3 \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_1^2 \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta \left(4 - \frac{1}{4}\right) d\theta$$

$$= \frac{15}{4} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{15}{8} \cdot \frac{\pi}{2} + \frac{15}{16} \sin 2\theta \Big|_0^{\frac{\pi}{2}}$$

$$-\frac{15}{32} \cos 2\theta \Big|_0^{\pi/2} = \frac{15}{16}$$

8.1

$$I_2 = \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \sin\theta \, dr \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin\theta \cdot \frac{r^3}{3}\Big|_1^2$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \cdot \frac{7}{3} \, d\theta$$

$$= \frac{7}{3} \left(-\cos\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{14}{3}$$

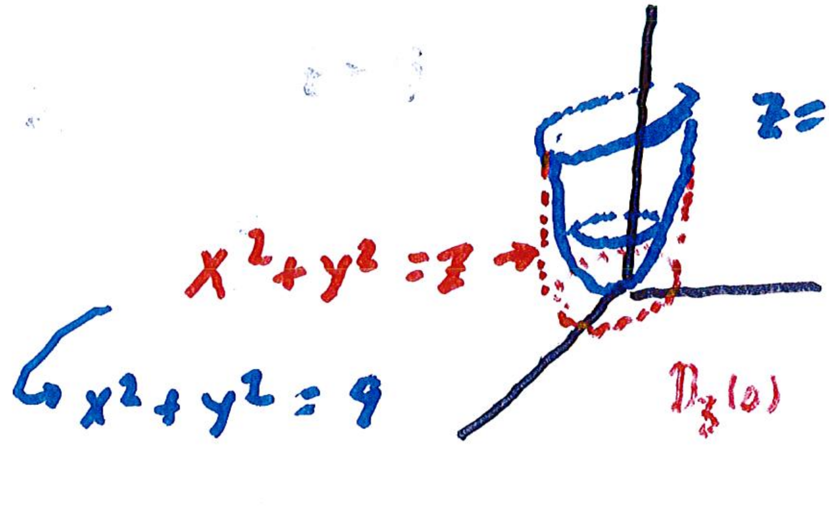
$$\boxed{\begin{array}{c} I_1 + I_2 \\ \shortparallel \\ \underline{\underline{\frac{15\pi}{16} + \frac{14}{3} + \frac{15}{16}}} \end{array}}$$

$$\therefore \iint_R = \frac{15\pi}{16} + \frac{14}{3} + \frac{15}{16}\cancel{\pi}$$

Ex. Find the volume of the

solid

bounded by $z = x^2 + y^2$ and $z = 9$



$$\text{Vol} = \int_0^{2\pi} \int_0^3 (9 - r^2)\, r \, dr \, d\theta$$

$$= 2\pi \int_0^3 9r - r^3 \, dr$$

$$= 2\pi \left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Big|_0^3$$

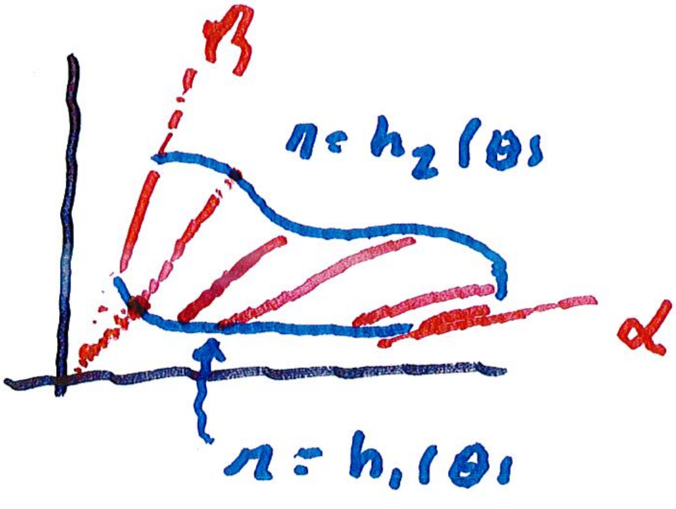
$$= 2\pi \left(\frac{9 \cdot 9}{2} - \frac{81}{4} \right)$$

$$= \frac{81}{4} \cdot 2\pi = \frac{81\pi}{2}$$

Now suppose D is bounded by

$$D: \{(r,\theta);\ \alpha \le \theta \le \beta\ ,$$

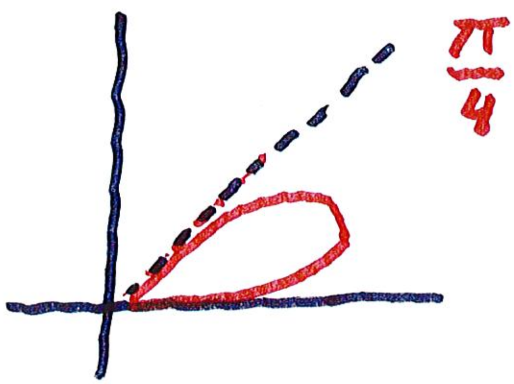
$$h_1(\theta) \le r \le h_2(\theta)$$



$$\iint_R f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Use a double integral to find the area in one loop of $r = \sin 4\theta$.

$0 < \theta < \frac{\pi}{4}$



$$A = \int_0^{\frac{\pi}{4}} \int_0^{\sin 4\theta} 1 \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/4} \left(\int_0^{\sin 4\theta} r \, dr \right) d\theta$$

$$= \int_0^{\pi/4} \cdot \frac{r^2}{2} \Big|_0^{\sin 4\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 4\theta}{2} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 8\theta}{2 \cdot 2} d\theta$$

$$= \frac{\pi}{16} - \frac{\sin 8\theta}{32} \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{16}$$

Ex Find area inside $r = 1 + \cos\theta$

Let A: area of top half

$$A = \int_0^{\cancel{\frac{\pi}{2}}\ \pi} \frac{1}{2}(h(\theta))^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left((1+\cos\theta)^2\right) d\theta$$

$$= \frac{\pi}{4} + \int_0^{\frac{\pi}{2}} \cos\theta + \frac{\cos^2\theta}{2} \, d\theta$$

$$\hookrightarrow = \sin\theta \Big|_0^{\pi/2} + \int_0 \frac{1 + \cos 2\theta}{}$$

$$= \frac{\pi}{4} + 1 + \int_0^{\pi/2} \frac{1 + \cos 2\theta}{4}\, d\theta$$

$$= \frac{\pi}{4} + 1 + \frac{\pi}{8} + \underbrace{\frac{\sin 2\theta}{8}\Big|_0^{\pi/2}}_{=0}$$

$$= 1 + \frac{3\pi}{8}$$

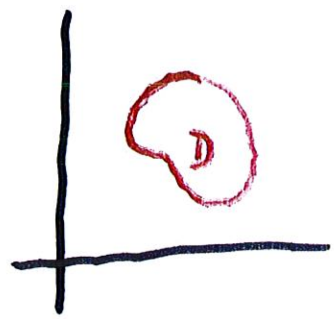
$\therefore$ Area $= 2 + \dfrac{3\pi}{4}$

15.4

~~15.5~~ – Density and Mass

Consider a thin plate in the shape of

a domain D.



The density of

D may vary

as a function of (x,y). We define

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A} \rightarrow \underline{\Delta m \approx \rho(x,y)\Delta A}$$

In the usual way, we can

approximate D by a union of

rectangles $R_{i,j}$, each of area

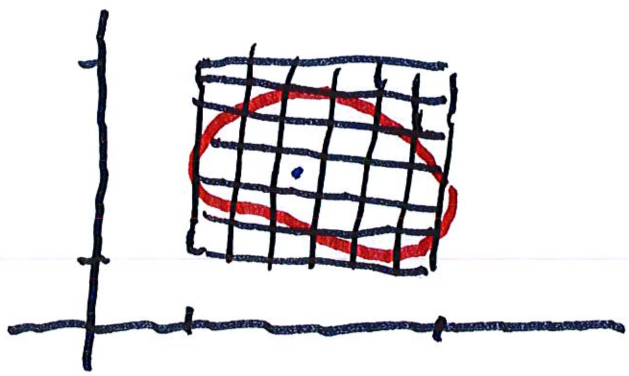
$\Delta A = \Delta x \, \Delta y$. The mass of

the rectangle R_{ij} is approx.

$= \rho(x_i, y_j) \, \Delta x \, \Delta y$.

$\therefore$ The total mass is approx

$$m \approx \sum_{i,j} \rho(x_i, y_j) \, \Delta x \, \Delta y$$



So $m = \iint_D \rho(x, y)\, dA$

Similarly, if $\sigma(x, y)$ = charge density at (x, y), then the total charge is

$$Q = \iint_D \sigma(x, y)\, dA$$

Moments and Center of Mass

In Chapter 8, we defined

Moment of D about y-axis $= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, \rho(x_i, y_j) \underbrace{\quad}_{dA}$

$$\rightarrow \iint_D x\, \rho(x,y)\, dA$$

$$= M_y$$

and

Mom. of D about x-axis $= \iint_D y\, \rho(x,y)\, dA$

$$= M_x$$

As before, the center of mass of

D is at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m},$$

where $m = \iint_D \rho(x,y)\, dA$.

Ex. Find the center of mass of D if

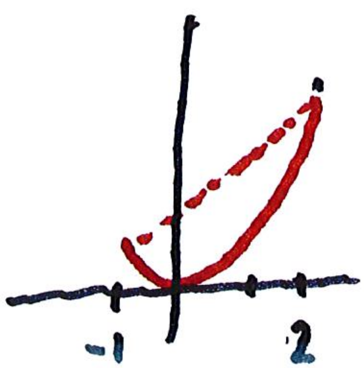
D is bounded by: $y = x^2$ and $y = x+2$

and $\rho(x,y) = y$.

$y = x^2$ and $y = x+2$ intersect

whe $x^2 = x+2$ or $x^2 - x - 2 = 0$

$\rightarrow (x-2)(x+1) = 0 \rightarrow x = -1$ or $x = 2$



$$\text{Mass} = \iint_D y \, dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} y \, dy \, dx$$

$$= \int_{-1}^{2} \frac{y^2}{2} \Big|_{x^2}^{x+2} dx = \int_{-1}^{2} \left(\frac{(x+2)^2}{2} - \frac{x^4}{2} \right) dx$$

$m =$

$$\bar{x} = \frac{M_y}{m} \qquad M_y = \iint_D x\,y\,dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} x y \, dy\, dx$$

$$= \int_{-1}^{2} \left. \frac{x y^2}{2} \right|_{y=x^2}^{y=x+2}$$

$$= \int_{-1}^{2} \frac{x(x+2)^2}{2} - \frac{x^5}{2} \, dx$$

$$M_y = \iint_D x\,y\,dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} x\,y\,dy\,dx$$

$$= \int_{-1}^{2} x\,\frac{y^2}{2}\Bigg|_{x^2}^{x+2} dx$$

$$\boxed{M_y = \frac{1}{2}\int_{-1}^{2} x(x+2)^2 - x^5\,dx}$$

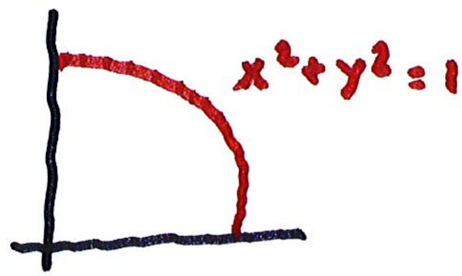
$\bar{y} = \frac{M_x}{m}$, where

$$M_x = \iint_D y^2 \, dA$$

$$= \int_{-1}^{2} \frac{y^3}{3} \Big|_{x^2}^{x+2} dx$$

$$\boxed{M_x = \frac{1}{3}\int_{-1}^{2} (x+2)^3 - x^6 \; dx}$$

Ex. Find $(\bar{x}, \bar{y})$ if D = region in first ~~octant~~ quadrant bounded by $x^2+y^2=1$ and where the density $= y$



$$\bar{x} = \frac{M_y}{m}$$

$$m = \iint_D y \, dA = \int_0^{\pi/2} \int_0^1 r\sin\theta \; r\,dr\,d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= -\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = \frac{1}{3}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\theta \, r\sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta\cos\theta}{4} \, d\theta$$

$$= \frac{1}{4} \left. \frac{\sin^2\theta}{2} \right|_0^{\frac{\pi}{2}} = \frac{1}{8}$$

$$\therefore \hat{x} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}$$

$$M_x = \iint_D y \cdot y \, dA$$

$$= \int_0^{\pi/2} \int_0^1 (r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} 1-\cos 2\theta \quad d\theta$$

$$= \frac{\theta}{8} - \frac{\sin 2\theta}{16}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{16}$$

$$\therefore \bar{y} = \frac{M_x}{m} = \frac{\pi}{16}\cdot 3 = \frac{3\pi}{16}$$