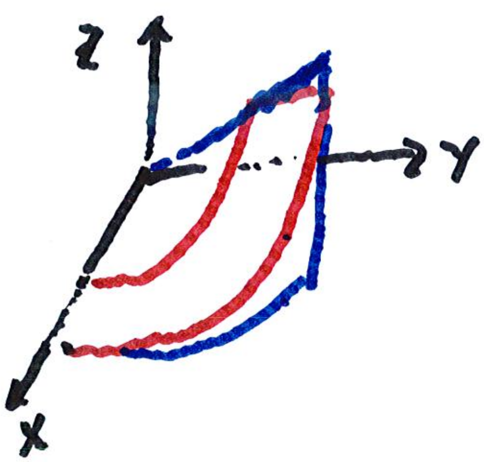
Triple Integrals in Spherical Coord.

Ex. Compute $\iiint_E z\,dV$ where E

is in the first octant and lies

between $x^2+y^2=1$ and $x^2+y^2=4$

and lies below $z=y$



$$\iiint_E z\,dV$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \int_0^{r\sin\theta} z \quad dz\; r\,dr\; d\theta$$

$\theta \qquad r \qquad z$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{z^2}{2} \Big|_0^{r\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r^3 \sin^2\theta}{2}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{8} \sin^2\theta \Big|_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 - \frac{1}{4}\right) \frac{\sin^2\theta}{2}\,d\theta$$

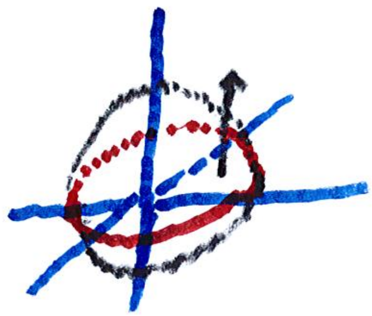
$$= \frac{15}{8} \int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{15}{8} \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{15\pi}{16} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta \; d\theta$$

$$= \frac{15}{32}\pi - \left(\frac{15 \sin 2\theta}{32} \Bigg|_0^{\frac{\pi}{2}} \right)$$

$$= \frac{15\pi}{32}$$

Ex. Let E be the region in the first octant bounded by $x^2 + z^2 = 4$ and bounded between $y = 4$ and $y = x$.



$0 < \rho < a$

$0 < \phi < \pi$

$0 < \theta < 2\pi$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \cdot \rho \cos\phi \cdot \rho^2 \sin\phi$$

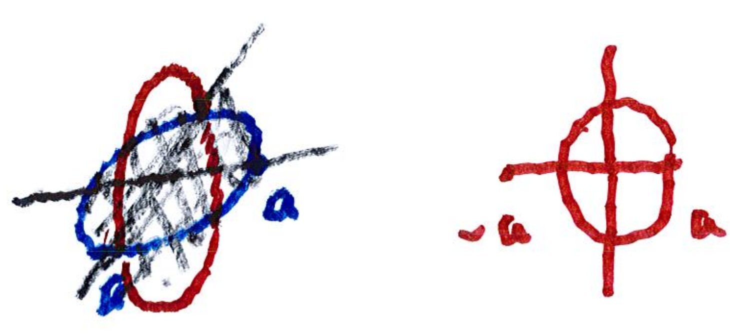
$$= 2\pi \cdot \left(\rho^5 \Big|_0^a \right) \int_0^{\pi} \sin\phi \cos\phi \, d\phi$$

$$= 2\pi \frac{a^6}{6} \cdot \frac{1}{2} \sin^2\phi \Big|_0^{\pi}$$

$$= \frac{a^6 \pi}{6} \cdot 0 = 0$$

Ex. Convert to spherical coordinates to compute the integral.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3)\, dz\, dx\, dy$$



circle in

5.1

$$\int_{-a}^{a}\int_{-\sqrt{a^2-}}^{\sqrt{a^2-x^2}} g(x,y)\,dx\,dy \quad = \text{integral of } g(x,y) \text{ on disk } D_a$$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}$$

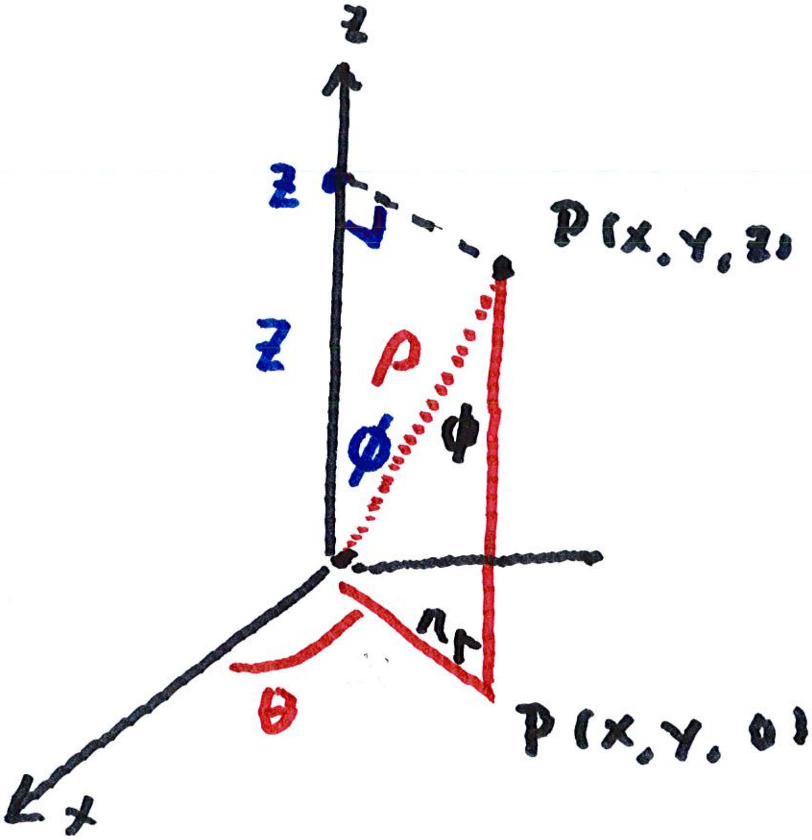
$$(x^2 z + y^2 z + z^3) = \rho^3\cdot\cos\phi\cdot\underbrace{\rho^2\sin\phi}_{d\phi\, d\theta\, d\rho}$$

$$= \int_0^{2}\int_0\int_0^{a} \rho^5\cos\phi\,\sin\phi\;d\theta\,d\phi\,d\rho$$

$$= 0,\quad \text{since } \int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\,d\phi = 0.$$

Spherical Coord.

ϕ : angle between ρ and positive z-axis



$$\frac{z}{\rho} = \cos\phi$$

$$z = \rho \cos\phi$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{r}{\rho} = \sin\phi \rightarrow r = \rho\sin\phi$$

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

Spherical Coord (ρ, θ, ϕ)

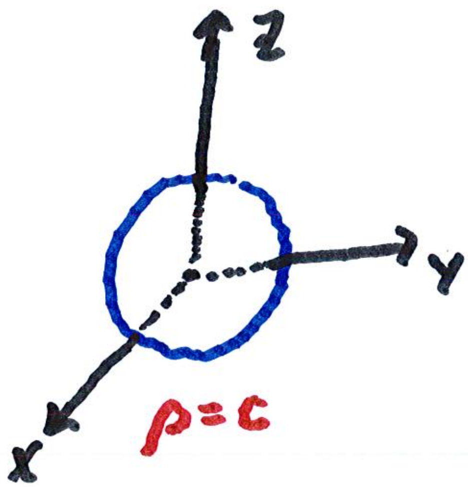
$$\rho = \sqrt{x^2+y^2+z^2} \quad \text{or} \quad \rho^2 = x^2+y^2+z^2$$

If we project (x, y, z) down to

the xy-plane, then

θ = usual coordinate of $(x, y, 0)$

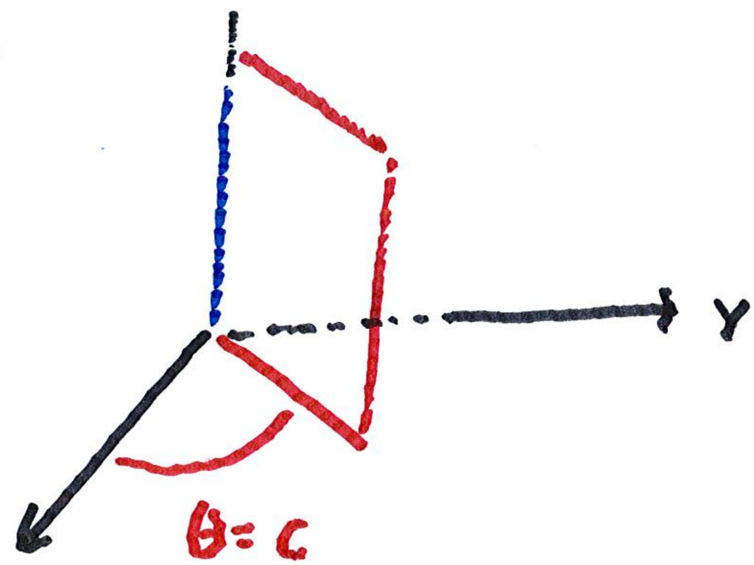
$$x = r\cos\theta \;, \; y = r\sin\theta$$



$\rho = c$

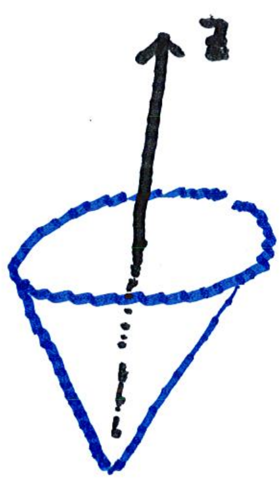
sphere

(ρ, θ, ϕ) are spherical coord:

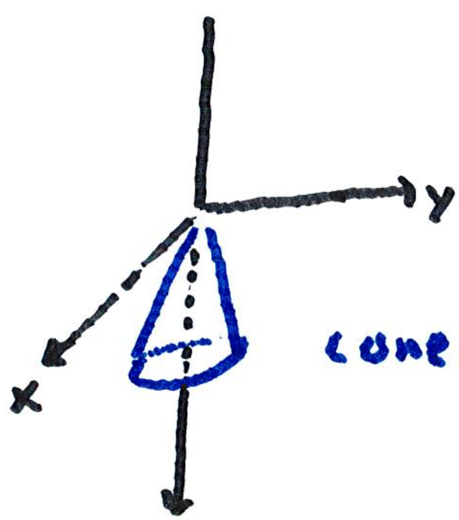


$\theta = c$

half-plane

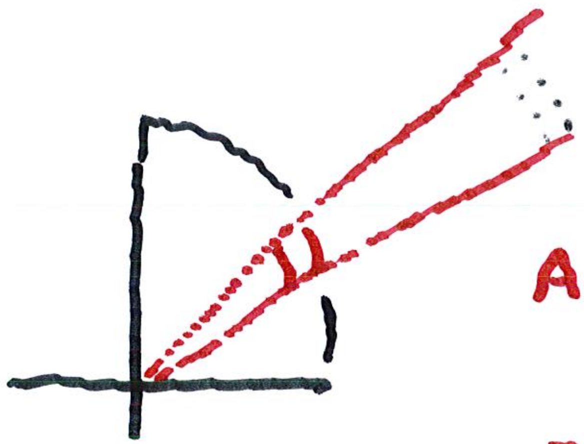


$\phi = c < \frac{\pi}{2}$ cone



$\frac{\pi}{2} < \phi < \pi$

In polar coord



Area of $\Delta_r \Delta_\theta$

$= r \Delta_r \Delta_\theta$

In spherical coord.

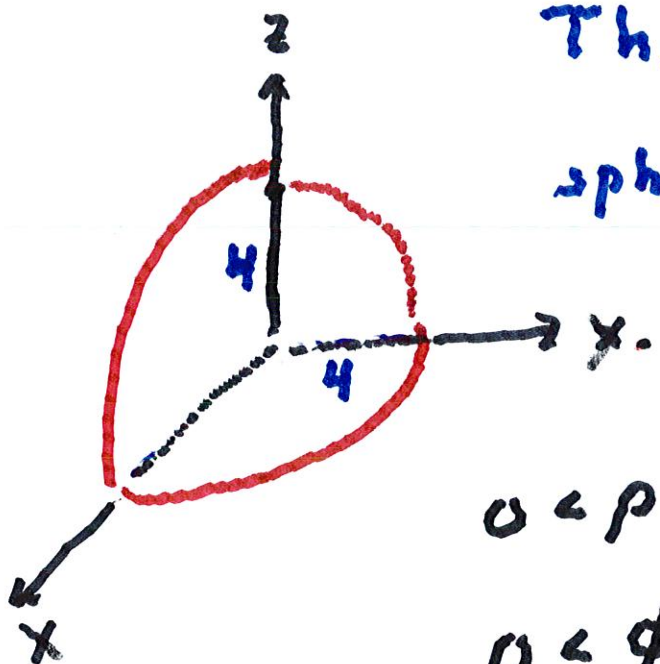
Eucl. Volume of $(\Delta_\rho \Delta_\theta \Delta_\phi)$-box

$$= \left(\rho^2 \sin\phi\right)\left(\Delta_\rho \Delta_\theta \Delta_\phi\right)$$

$\therefore$ When converting from (x, y, z) integral to a spherical, You must include $\rho^2 \sin\phi$.

Ex. Let E = region of ball (about $(0,0,0)$) of radius 4 in the first octant.

Compute $\displaystyle\iiint_E z \, dV$.



This is a 'box' in spherical coord.

$$0 < \rho < 4$$

$$0 < \phi < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{2}$$

$$\iiint_E z\, dV = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho\cos\phi \;\rho^2 \sin\phi \; d\phi\, d\rho\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi \left.\frac{\rho^4}{4}\right|_0^4 d\phi\, d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi \, d\phi\, d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \left.\frac{1}{2}\sin^2\phi\right|_0^{\frac{\pi}{2}} d\theta$$

$$= 32 \int_0^{\frac{\pi}{2}} d\theta = 16\pi$$

Find the center of mass of

$$E = \{ (x, y, z) \quad 1 \le x^2 + y^2 + z^2 \le 4,$$
$$\text{where } z \ge 0 \}$$

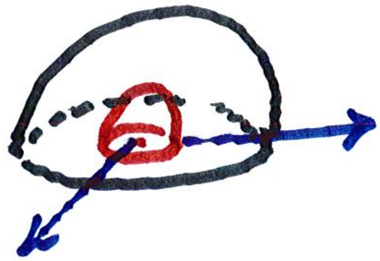
Assume density = 1

$1 \le \rho^2 \le 4 \rightarrow 1 \le \rho \le 2$

$z \ge 0 \rightarrow \quad 0 \le \phi \le \frac{\pi}{2}$

$$\text{mass} = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 1 \cdot \rho^2 \sin\phi$$

$\theta \quad \phi \quad \rho$



$$\iiint_E = 2\pi \int_0^{\frac{\pi}{2}} \int_1^2 \rho^2 \sin\phi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\phi \; \frac{\rho^3}{3} \Big|_1^2$$

$$= \frac{2\pi}{3} \cdot (8-1) \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi$$

$$= -\frac{14\pi}{3} \int_0^{\frac{\pi}{2}} -\overset{\cos\phi}{\cancel{\sin\phi}} \, d\phi$$

$$= \frac{-14\pi}{3} \cos\phi \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{-14\pi}{3} (0 - 1) = \frac{14\pi}{3}.$$

$$M_{xy} = \iiint_E z \, dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \rho\cos\phi \; \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\rho^4}{4} \sin \overset{\sin^2\phi/2}{\phi} \Big|_1^2 \, d\phi$$

$$= \frac{\pi}{2} (16-1) \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi$$

$$= \frac{15\pi}{2} (-\cos\phi) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{15\pi}{2} (0 - (-1)) = \frac{15\pi}{2}$$

$$\bar{z} = \frac{M_{xy}}{\frac{14\pi}{3}} = \frac{\frac{15\pi}{2}}{\frac{14\pi}{3}}$$

$$= \frac{45}{28} \qquad \bar{y} = 0 \text{ and } \bar{x} = 0$$

by symmetry

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{45}{28}\right)$$

Ex. Find the volume of the region below the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{x^2+y^2}$

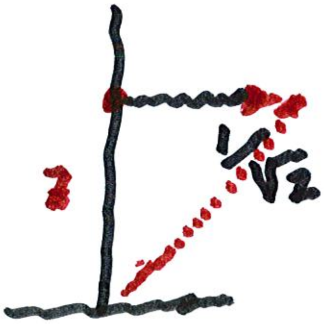
Set $r=\sqrt{x^2+y^2}$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

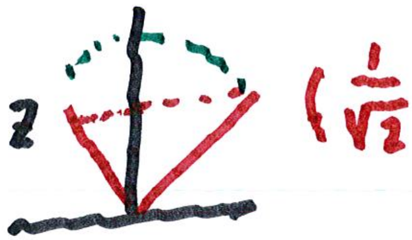
$z^2 = 1-r^2$

and $z=r \to z^2=r^2$

$\therefore \quad 1-r^2 = r^2 \to 2r^2=1 \to r=\frac{1}{\sqrt{2}}$

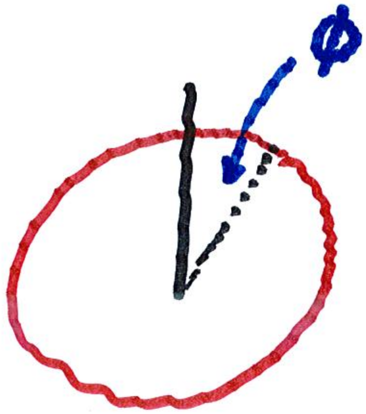


$z = r = \frac{1}{\sqrt{2}}$



$\therefore \tan\phi = \dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$\Rightarrow \phi = \frac{\pi}{4}$



$$V = \int_{\underset{\theta}{0}}^{2\pi} \int_{\underset{\phi}{0}}^{\pi/4} \int_0^1 1 \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{\rho^3}{3}\Big|_0^1 \int_0^{\frac{\pi}{4}} \sin\phi \; d\phi \qquad \cancel{= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right.}$$

$$= \frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\text{Vol} = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^a \; d\rho \, d\phi \, d\theta$$

$$\rho^2 \sin\phi$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\phi \int_0^a \cancel{\rho^2} \, \rho^2 \, d\rho \; d\phi$$

$$= 2\pi \cdot \frac{a^3}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin\phi \, d\phi$$

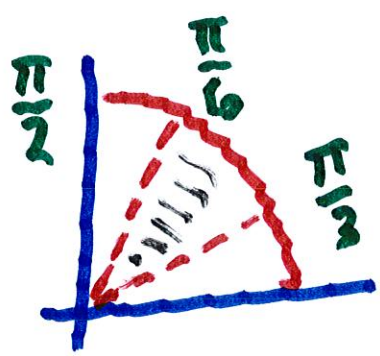
$$= \frac{2\pi a^3}{3} \left(-\cos\phi \right) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{2\pi a^3}{3} \left(-\frac{1}{2} - \left\{ -\frac{\sqrt{3}}{2} \right\} \right.$$

In other words

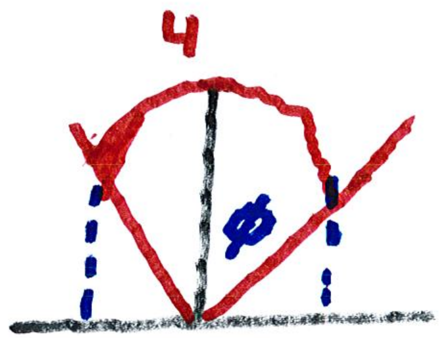
$$dV = \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

Ex. Express the volume ~~of the part of~~ the part of the ball $\rho \le a$ that lies between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$



$$= \frac{\pi a^3}{3}\left(\sqrt{3}-1\right)$$

Ex Find the volume of the solid that lies ~~between the~~ within the sphere $x^2+y^2+z^2=4$, above the xy-plane and below the cone $z=\sqrt{x^2+y^2}$



$$\frac{\pi}{4} < \phi < \frac{\pi}{2}$$

$$\text{Vol} = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \frac{\rho^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \, d\phi$$

$\rho = 2$

$$= \frac{2\pi\rho^3}{3} \left(-\cos\phi \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right)$$

$$= \frac{2\pi 8^3}{3} \left(0 + \frac{1}{\sqrt{2}} \right)$$

$$= \frac{16\pi}{3} = 8\pi\sqrt{2}$$