16.2 Line Integrals.

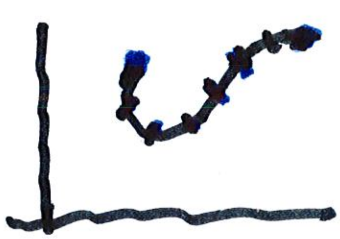
Suppose we are given a curve C

defined by $\{x(t), y(t)\}$ for $a \le t \le b$.

We want to define the integral

$$\int_C f(x,y)\,ds$$ of a function $f(x,y)$

that is defined for all (x,y) on C.



As usual, we partition the curve by defining points

$(x_i, y_i) = (x(t_i), y(t_i))$,

where $a < \dots < t_{i-1} < t_i < \dots t_n = b$.

The i-th segment has length approximately =

$$\ell_i = |\langle x'(t_i), y'(t_i) \rangle| \Delta t ,$$

To define $\int_C f(x,y)\,ds$,

we multiply ℓ_i by $f(x_i, y_i)$

Thus, we obtain

$$\int_C f(x,y)\,ds \approx \sum_{i=1}^{n} f(x_i, y_i)\cdot \overbrace{|\langle x'(t_i), y'(t_i)\rangle|\,\Delta t}$$

Letting $n \to \infty$, we get

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))$$

$$|\langle x'(t), y'(t)\rangle|\,dt$$

Let S = straight path from

$\langle 1,2\rangle$ to $\langle 4,8\rangle$. Then

we get $\int_S (x^2 + y)\,ds$ is par. by

$\vec{r}(t) = \langle 1, 2 \rangle + t\langle 3, 6 \rangle, \quad 0 \le t \le 1.$

$\therefore \; x = 1 + 3t, \quad y = 2 + 6t$

$\langle x', y' \rangle = \langle 3, 6 \rangle$

$|r'(t)| = |\langle 3, 6 \rangle| = \sqrt{45}$

$$\int_0^1 (x^2 + y)\sqrt{45}\, ds$$

$$= \int_0^1 (\underbrace{1 + 6t + 9t^2}_{} + \underbrace{2 + 6t}_{y})\underbrace{\sqrt{45}\, dt}_{ds}$$

$$= \left(t + 3t^2 + 3t^3 + 2t + 3t^2 \right) \Big|_0^1$$

$$= (1 + 3 + 3 + 2 + 3) = 12$$

Given a curve C parameterized by $x(t)$ and $y(t)$, the integral is

$$\int_a^b f\big(x(t), y(t)\big) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{|\vec{r}\,'(t)|} \, dt$$

When $f(x,y) = 1$, then

$$\int_C f(x,y)\,ds$$ becomes

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt ,$$

which is the length of L.

Ex. Let C be the portion of the unit circle in the first quadrant.

Compute $\displaystyle\int_C 4x^2 - 2x^4 y \ ds$.

Since $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt$

$= \sqrt{(-\sin t)^2 + (\cos t)^2} \ dt$

$= 1 \cdot dt$, the integral becomes

$$\int_0^{\pi} 4\cos^2 t - 2\cos^4 t \sin t \ dt$$

$$= \int_0^{\pi} 4\left(\frac{1+\cos 2t}{2}\right) + 2\cos^4 t \sin t \ dt$$

$$= \int_0^{\pi} 2 + 2\cos 2t + 2\cos^4 t \sin t \, dt$$

$t = \pi$

$$= \left[2t + \sin 2t - \frac{2\cos^5 t}{5} \right]_{t=0}^{t=\pi}$$

$$= 2\pi + 0 + \frac{1}{5} + \frac{1}{5} = 2\pi + \frac{2}{5}$$

We can describe the integral of a function along a curve by

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

dt

The value of the line integral does not depend on the parameterization, provided that the curve is traversed exactly once as t increases

from a to b.

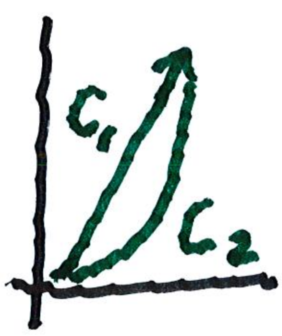
If C is a union of a finite

number of curves (that are

piecewise smooth.)

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots$$

$$\cdots + \int_{C_n} f(x,y)\,ds$$



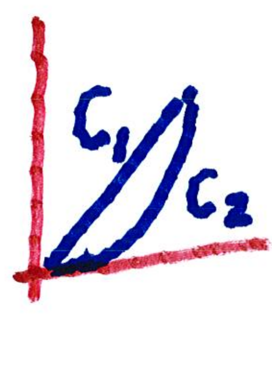
C_1 : $y = 2x$, $0 \le x \le 2$

C_2 : $y = x^2$, $0 \le x \le 2$

For example, if C_1 is the line segment from $(0,0)$ to $(1,1)$ and C_2 is the curve $y = x^2$ from $(0,0)$ to $(1,1)$, then

$$\underbrace{\int_{C_2} xy\, ds} + \underbrace{\int_{C_1} xy}, \quad \text{then}$$

$$C = C_1 \cup C_2.$$



$$= \cancel{\int_{-2}^{2} (4-t^2)\sqrt{1^2+4t^2}}\;dt,$$

We can replace Δs_i by

$\Delta x_i = x_i - x_{i-1}$ or

$\Delta y_i = y_i - y_{i-1}$.

Then one obtains

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

or $$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Line Integrals in Space

Suppose a curve C is given by

$x = x(t)$, $\quad y = y(t)$, $\quad z = z(t)$.

Given a fcn. $f(x, y, z)$, we obtain

$$\int_C f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

This can be written more compactly

as $\displaystyle\int_a^b f(\vec{r}(t))\, |r'(t)|\, dt$

We get a different line integral when we replace Δs_i by $x_i - x_{i-1}$ or Δs_i by $y_i - y_{i-1}$.

This gives

$$\int_C f(x,y)\,dx = \int_C f(x(t), y(t))\, x'(t)\,dt$$

or $$\int_C f(x, y(t))\,dy = \int_C f(x(t), y(t))\, y'(t)\,dt$$

Ex. Evaluate the line integral

$$\int_C y\,dx - 3x\,dy ,$$

where $x = 2+3t$, $y = 1-4t$, $0 \le t \le 1$

and $x' = 3$, $y' = -4$

$$\therefore \int_C (1-4t)(3) - 3(2+3t)(-4) \quad 0 \le t \le 1$$

$$= \int_0^1 (17+24) + 36t \; dt$$

$$= 41 + 18 = \underline{\underline{59}}$$

Evaluate the line integral

$$\int_C y\,dx - 3x\,dy \quad , \quad \text{with}$$

C defined by

$x = 2+3t$, $\quad y = 1-4t$

for $0 \leq t \leq 1$.

$$\int_C (1-4t)(3) - 3(2+3t)(-4)$$

$(0 \leq t \leq 1)$

$$= \int_0^1 27 + 12t \; dt =$$