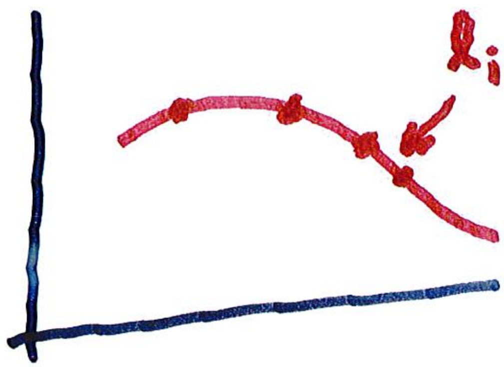
16.2 Line Integrals cont'd.

We learned that there are

two kinds of line integrals

$$\int_C f(x,y) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$



ℓ_i = length of i-th segment

This is an <u>integral of a function</u>

Suppose that $\vec{r}(t) = (x(t), y(t))$

Then we define $\int_C f(x,y)\,ds$

by $\int_C f(x,y)\,ds$

$$= \int_a^b f(x(t), y(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

where $\vec{r}(t) = (x(t), y(t))$

Ex. Let $x(t) = 2\cos t$, and

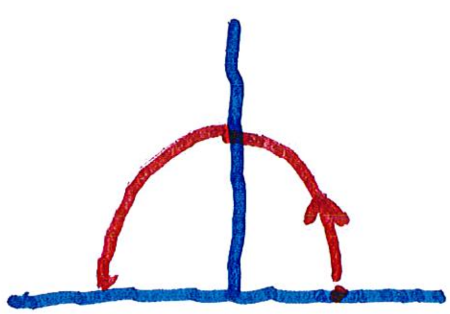
$y(t) = 2\sin t$ for $0 \le t \le \pi$.

Thus $\vec{r}(t) = (2\cos t, 2\sin t)$

is a parametrization of

the half-circle of radius 2

in the upper halfplane



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Ex. Evaluate $\int_C (2xy - x^2)\, ds$

$$= \int_0^{\pi} 8\cos t \sin t - 4\cos^2 t \; dt$$

$$= \int_0^{\pi} 8 \sin t \cos t - 2(1 + \cos 2t) \; dt$$

$$= \Big[4\sin^2 t - 2t - \sin t \Big]_{t=0}^{t=\pi}$$

$$= (0 - 2\pi - 0) - (0 - 0 - 0)$$

$$= \underline{\underline{-2\pi}}$$

$$C = \{ (\cos t, \ \sin t, \ 3t) : 0 \le t \le \pi \}$$

$$\vec{r}'(t) = (-\sin t, \ \cos t, \ 3)$$

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 3^2}$$

$$= \sqrt{10}$$

$$\therefore \int_C = \int_0^{\pi} (3\cos t - \sin t + 3t) \sqrt{10} \, dt$$

$$= \left(3\sin t + \cos t + \frac{3t^2}{2} \Big|_0^{\pi} \right) \sqrt{10}$$

$$= \left[\left(-1 + \frac{3\pi^2}{2} \right) - 1 \right] \sqrt{10}$$

Another kind of line integral

is $\int_C f(x,y)\,dx$ or $\int_C g(x,y)\,dy$

Note that if $x = x(t)$, then

$$dx = x'(t)\,dt,$$

and similarly, if $y = y(t)$,

then $dy = y'(t)\,dt$

Suppose that $C = \{(t^2, 3t) : 0 \le t \le 1\}$

Compute $\displaystyle\int_C xy^2\,dx$.

$x = t^2$, and $y = 3t$, and $x'(t) = 2t$

$$\therefore \int_C xy^2\,dx = \int_0^1 t^2 (3t)^2 \cdot 2t\,dt$$

$dx = 2t\,dt$

$$= \int_0^1 18 \cdot t^5\,dt$$

$$= \frac{18}{6} = 3$$

Now we compute

$$\int_C x^2 y \, dy \quad \text{where}$$

C is the straight segment from $(0,1)$ to $(2,3)$

Then $\vec{r}(t) = (0,1) + t(2,2)$

$\therefore x(t) = 2t$

$y(t) = 1 + 2t$

and $y'(t) = 2$

We obtain the integral

$$\int_0^1 (2t)^2 (1+2t)\cdot \underbrace{2\,dt}_{dy = y'(t)\,dt}$$

$$= \int_0^1 8t^2 (1+2t)\,dt$$

$$= \int_0^1 8t^2 + 16t^3 \quad dt$$

$$= \frac{8}{3} + 4 = \frac{20}{3}$$

We usually combine the integrals:

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

$$= \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Ex. Evaluate $\int_C y^3\,dx - x^2\,dy$,

where C is parameterized by

For a curve in space, suppose that $x = x(t)$, $y = y(t)$, $z = z(t)$, for $a \le t \le b$.

$$\int_C f(x,y,z)\,ds$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

One can also define expressions such as

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

where one multiplies each term

by $x'(t)$, $y'(t)$, and $z'(t)$.

Now we define line integrals

of a vector field.

Recall that if $\vec{F}$ is a vector,

and if is a unit vector.

then $\vec{F} \cdot \vec{v}$ is the component of $\vec{F}$ in the $\vec{v}$-direction. For example, if $\vec{F} \in \mathbb{R}^3$, then

$$\vec{F} \cdot \langle 0, 1, 0 \rangle = F_2 \text{ , and}$$

$$\vec{F} \cdot \langle 0, 0, 1 \rangle = F_3 \text{ , and}$$

$$\frac{1}{\sqrt{2}} F_1 + \frac{1}{\sqrt{2}} F_2 \text{ is}$$

the component of $\vec{F}$ in the $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$ direction.

Now let C be a curve in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is defined by

$$\vec{r}(t) = \langle x(t), y(t) \rangle .$$

The velocity at time t is

$$\vec{r}'(t) = \langle x'(t), y'(t) \rangle .$$

The work done during a Δt_i time interval is

$$\vec{F}(x_i, y_i) \cdot \underbrace{\frac{\vec{r}\,'(t_i)}{|\vec{r}\,'(t_i)|}}_{= \vec{T}(t_i)} \underbrace{|\vec{r}\,'(t_i)|\,\Delta t_i}_{s_i}.$$

Letting $n \to \infty$, and summing up over all time intervals,

we get

$$W = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\, dt$$

Since $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$.

we get

$$W = \int_a^b P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)$$

This is the line integral of

the force $P(x,y)\vec{i} + Q(x,y)\vec{j}$

along a curve $C = (x(t), y(t))$

Ex. Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$\vec{F} = xy\,\vec{i} + yz\,\vec{j} + zx\,\vec{k}$, and

C is the twisted cubic

$x = t, \quad y = t^2, \quad z = t^3, \quad 0 \le t \le 1$

$$\int_0^1 t \cdot t^2 \cdot 1 + t^5 \cdot 2t + t^4 \cdot 3t^2 \; dt$$

$$= \frac{1}{4} + \frac{2}{7} + \qquad + \frac{3}{7} = \frac{1}{4} + \frac{5}{7}$$

$$= \frac{27}{28}$$

Ex. Compute $\int_C z^2\,dx + x^2\,dy + y^2\,dz$,

where C = line segment from

$(1,0,0)$ to $(4,1,2)$

$\vec{r}(t) = \langle 1,0,0\rangle + t\langle 3,1,2\rangle$.

$\therefore\ x(t) = 3t+1$

$y(t) = t$

$z(t) = 2t$

Then the integral is

$$\int_0^1 (4t^2)\cdot 3 + (3t+1)^2\cdot 1 + t^2\cdot 2t \; dt$$

$$= \int_0^1 12t^2 + 9t^2 + 6t + 1 + 2t^3 \, dt$$

$$= \frac{21}{3} + 3 + 1 + \frac{1}{2} = \frac{23}{2}$$

Note: When C is parameterized with the opposite orientation, the line integral $\int_{-C} \vec{F} \cdot d\vec{r}$, the sign changes by (-1)

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

Ex. Compute $\int_C x\,dx + y\,dy + xy\,dz$,

if C is defined by

$\vec{r}(t) = (\cos t)\vec{i} + \sin t\,\vec{j} + t\vec{k}$

for $0 \le t \le 1$.

$x = \cos t \quad y = \sin t \quad z = t$

$x' = -\sin t \quad y' = \cos t \quad z' = 1$

$$\int_C = \int_0^1 \cos t(-\sin t) + \sin t(\cos t) + 1 \, dt$$

$$= t\Big|_0^1 = \frac{1}{2}$$

$\sin t \cos t$

Ex. Compute $\int_C \vec{F} \cdot d\vec{r}$ if

$$\vec{F} = (x+y)\vec{i} + (y-z)\vec{j} + z^2\vec{k}$$

and $\vec{r}(t) = t^2\vec{i} + t^3\vec{j} + t^2\vec{k}$

$x(t) = t^2 \quad y(t) = t^3 \quad z(t) = \cancel{2t}\, t^2$

$x' = 2t \quad y' = 3t^2 \quad z' = 2t$

$$\int_C = (t^2+t^3)2t + (t^3-t^2)3t^2 + t^4 \cdot 2t$$

$$= \int_0^1 2t^5 - t^4 + 2t^3$$

$$= -\frac{2}{6} - \frac{1}{5} + \frac{2}{4} = \frac{-17}{60}$$

To sum up line integrals of vector fields:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz,$$

where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$= \int_a^b \Big(P(x(t), y(t), z(t))\, x'(t)$$

$$+ \int_a^b Q(x(t), y(t), z(t))\, y'(t)$$

$$+ \int_a^b R(x(t), y(t), z(t))\, z'(t) \Big)\, dt$$