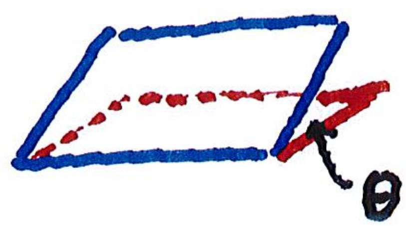
If 2 planes are not parallel, then they intersect in a straight line and the angle between them is defined as the acute angle between the normal vectors.



Ex. Find the angle between

$2x + y + z = 1$ and $x + 2y - z = -1$.

$\vec{n}_1 = \langle 2, 1, 1 \rangle$ $\quad\quad$ $\vec{n}_2 = \langle 1, 2, -1 \rangle$

$$\therefore \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2+2-1}{\sqrt{6}\,\sqrt{6}}$$

$$= \frac{3}{6} = \frac{1}{2} \qquad \therefore \theta = \frac{\pi}{3}$$

Ex. Express the line of intersection of the above using symmetric equations.

Note that the above line lies in both planes P_1 and P_2.

$\therefore$ If $\vec{v}$ is the direction vector of the line L, then $\vec{v}$ is $\perp$ to $\vec{n}_1$ and $\vec{v}$ is $\perp$ to $\vec{n}_2$.

$$\therefore \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= -3\vec{i} \; 3\vec{j} - 3\vec{k}.$$

We still need a point P in L.

Set $z = 0$ $\rightarrow$ $2x + y = 1$

$x + 2y = -1$

$\rightarrow x = 1, \; y = -1$

$\therefore$ $(1, -1, 0)$ is in L

$\therefore$ $\langle x, y, z \rangle = \langle 1, -1, 0 \rangle +$

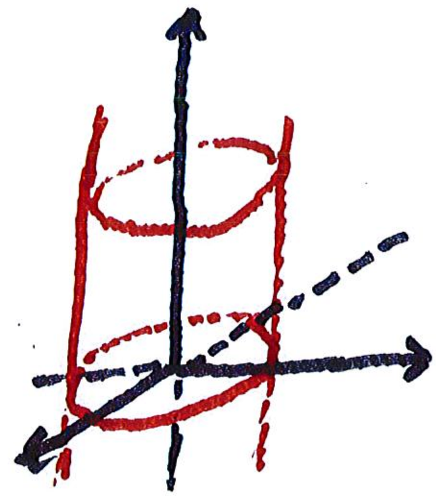
$+ t \langle -3, 3, 3 \rangle$

Sym. Equations are

$$\frac{x-1}{-3} = \frac{y+1}{3} = \frac{z}{3}$$

12.6 Surfaces in 3-dimensions

The cylinder is described by



$$x^2 + y^2 = r^2$$

Note that the equation is independent of z.

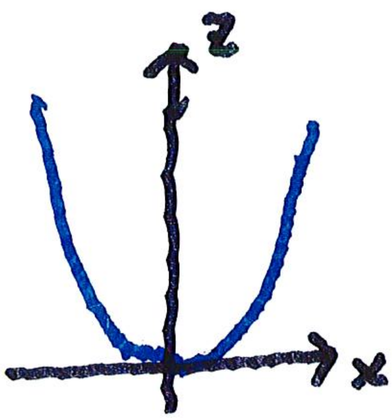
Thus z can assume any value.

Ex. Sketch the surface

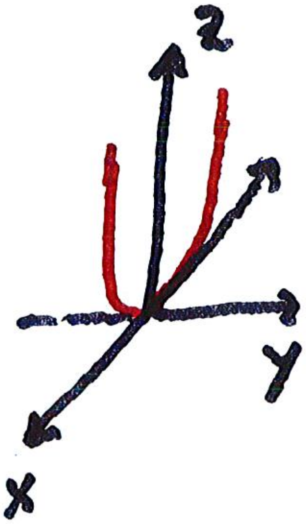
(ind. of y)

$z = x^2$

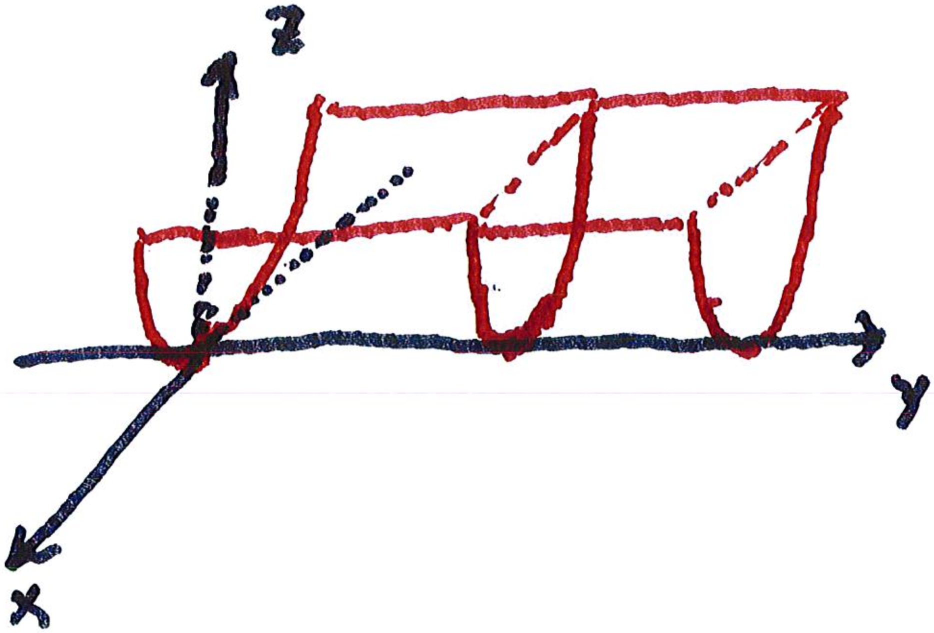
First draw the curve in the xz-plane.



Then draw curve in the xz-plane in 3 dim.

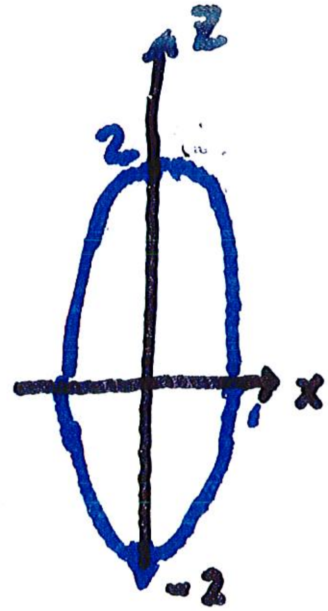


Then slide the curve in y-direction



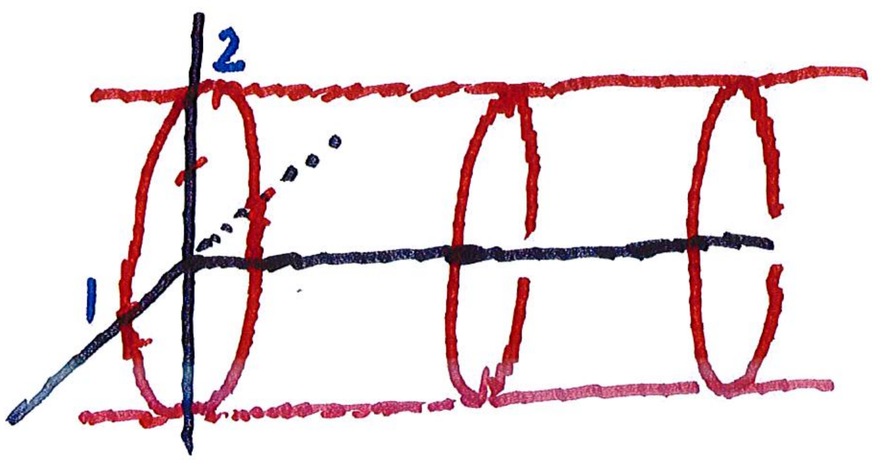
Ex Sketch the cylinder

$$x^2 + \frac{z^2}{4} = 1$$



The equation is independent of y.

Now slide the ellipse in the y- direction



Quadric Surfaces.

This is the graph of second-degree equations satisfying

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

By making translations and rotations, this can be put in the form

I $\quad Ax^2 + By^2 + Cz^2 + J = 0$ or

II $\quad Ax^2 + By^2 + Iz = 0$

Ex. Suppose in type I that

A, B, and C are all positive.

Ex. $\quad x^2 + \frac{y^2}{4} + \frac{z^2}{16} = 1$.

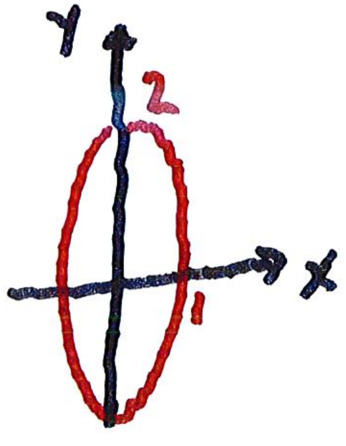
This is an ellipsoid.

We can write this as

$$x^2 + \frac{y^2}{4} = 1 - \frac{z^2}{16}$$

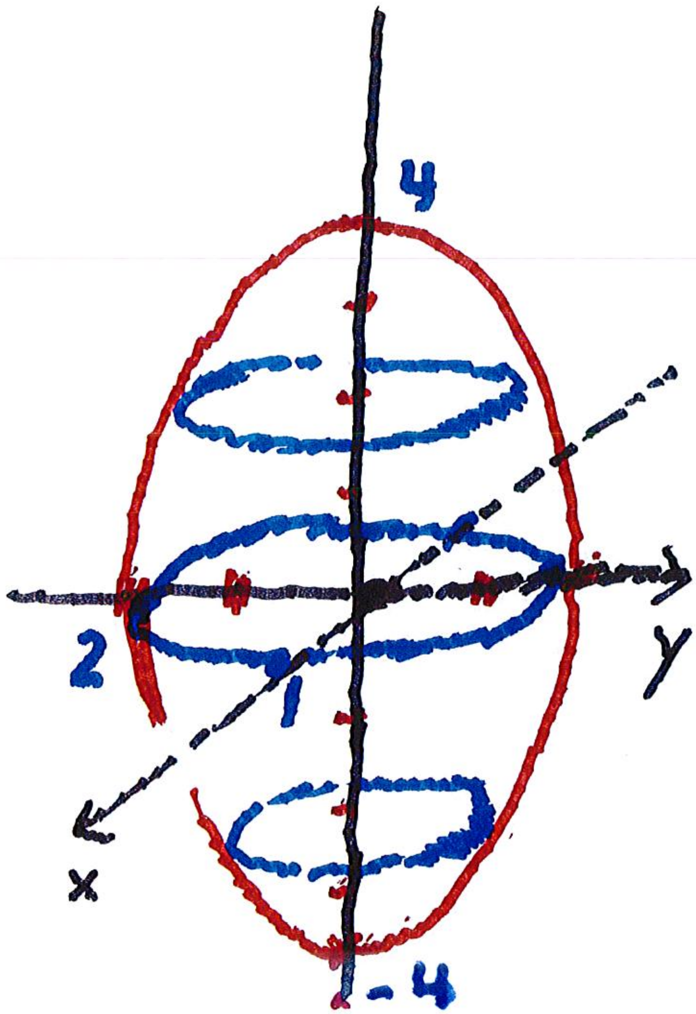
When $z = 0$, the "trace"

is $\frac{x^2}{1} + \frac{y^2}{4} = 1$



As z increases

$1 - \frac{z^2}{16}$ decreases



The ellipses are traces for different values of z $(-4 \leq z \leq 4)$

The trace is the cross-section obtained by fixing one of the variables.

We often analyze a surface by finding various traces.

Ex. Now consider the elliptic paraboloid.

$$z = x^2 + \frac{y^2}{4}$$

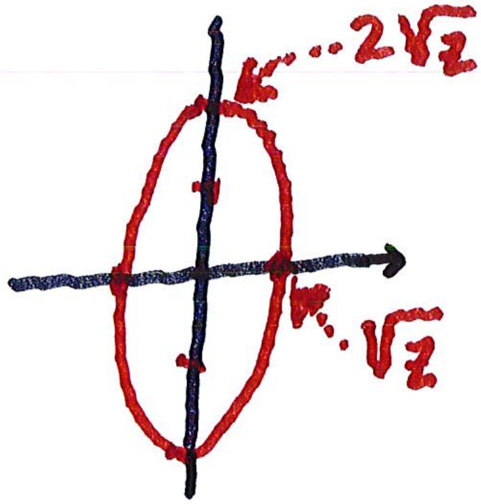
If $z = 0$, then $x = 0$, $y = 0$

If $z > 0$,

$$(\sqrt{z})^2 = x^2 + \frac{y^2}{4}$$

$$\rightarrow \quad 1 = \left(\frac{x}{\sqrt{z}}\right)^2 + \left(\frac{y}{2\sqrt{z}}\right)^2$$

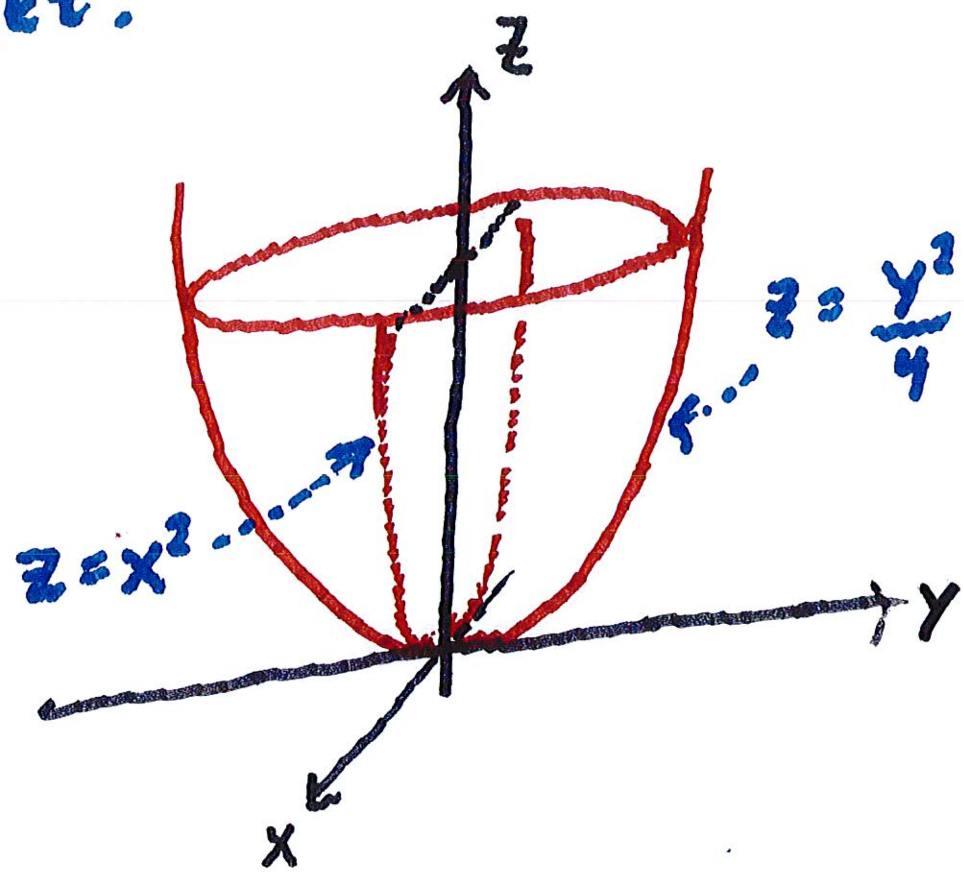
This is an ellipse



, which gets bigger as $\sqrt{z}$ increases.

If $y=0$, the trace is $z=x^2$, and if $x=0$, the trace is $z=\frac{y^2}{4}$

We get:



Ex. The hyperbolic paraboloid

is given by $z = y^2 - x^2$:

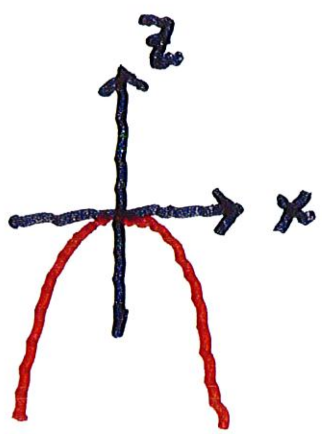
For fixed y, $z = y^2 - x^2$

(if $y = 0$) $z = -x^2 + y^2$

Ex. The hyperbolic paraboloid is given by

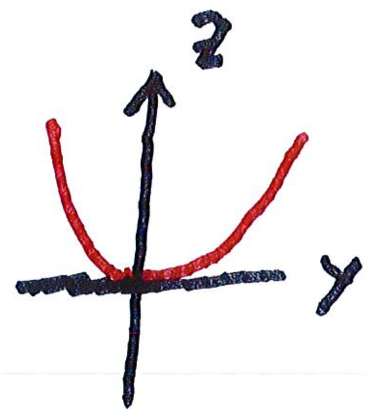
$z = y^2 - x^2$

For $y=0$ the surface is given by $z = -x^2$

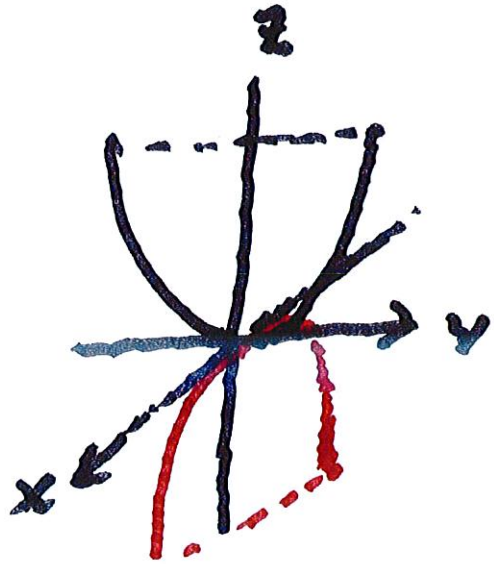


and if $x=0$,

and the trace of $x = 0$ is $z = y^2$

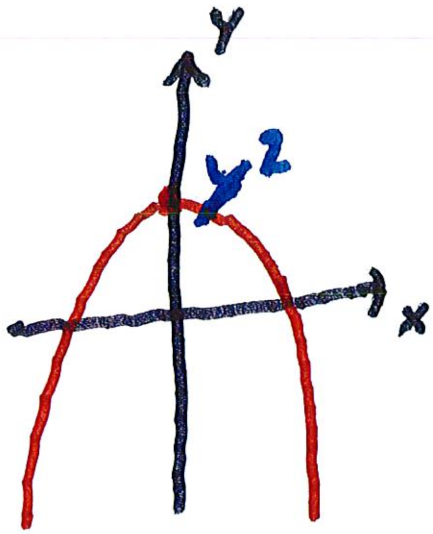


In 3 dimensions:



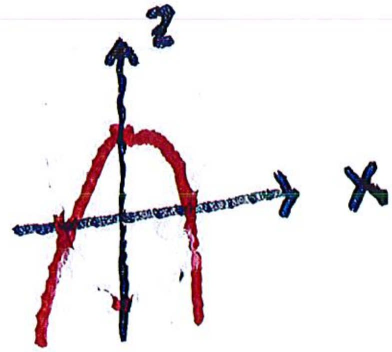
More generally, for fixed y,

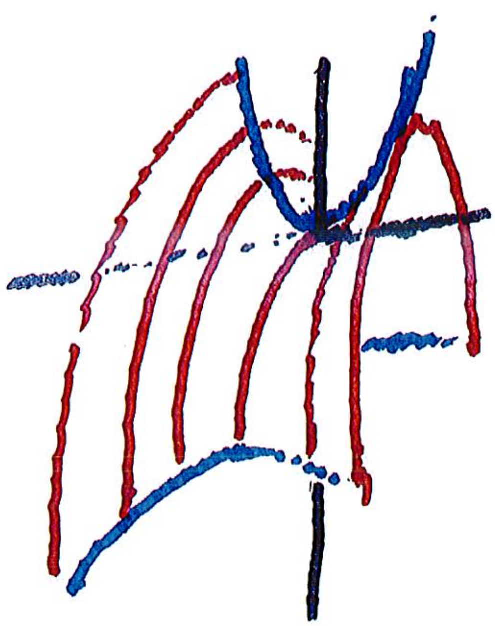
the curve is $z = y^2 - x^2$



The general surface is:

For fixed y, $z = y^2 - x^2$





The origin $(0,0,0)$ is a saddle point

Ex. Now consider the

equation $\qquad\qquad\qquad\qquad < 0$

$$x^2 + y^2 - z^2 = -4$$

or $x^2 + y^2 = z^2 - 4$

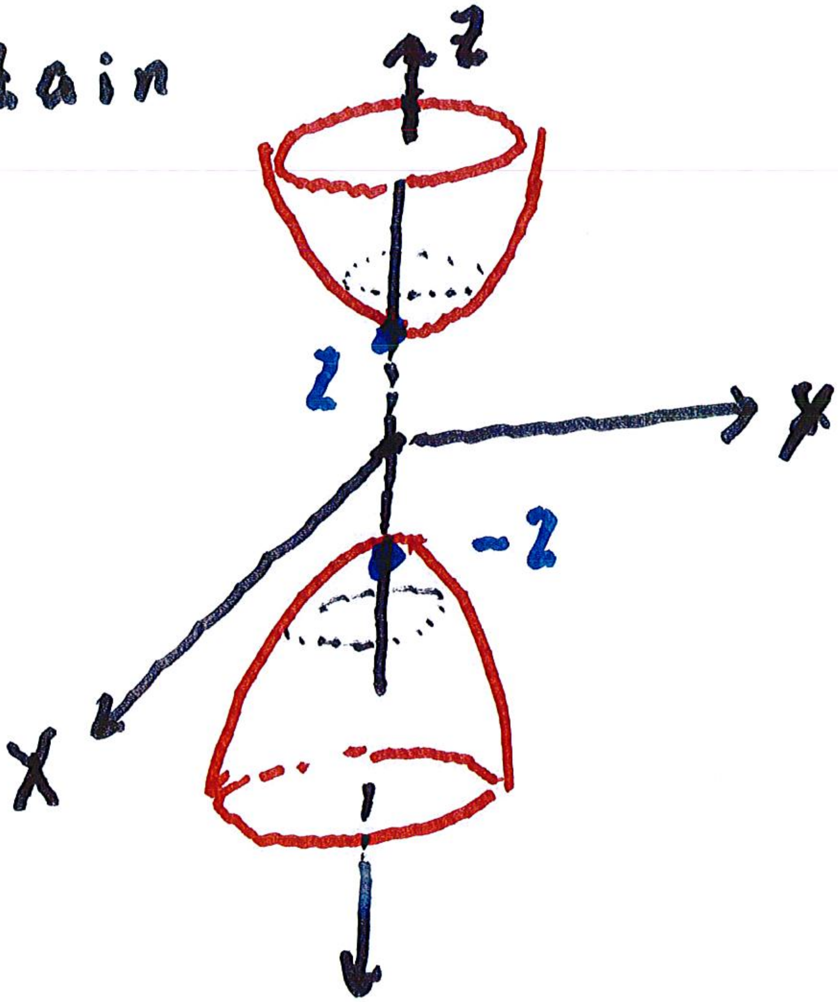
There is no solution

if $z^2 < 4$. If $z^2 \geq 4$

we have a circle of

radius of $\sqrt{z^2-4}$.

We obtain



This is a "hyperboloid

of 2 sheets

Now suppose that

2 of the coefficients

A, B, C are positive

and the other is negative.

> 0

Ex. $x^2 + y^2 - z^2 = 4$

or $x^2 + y^2 = 4 + z^2$

For fixed z, the

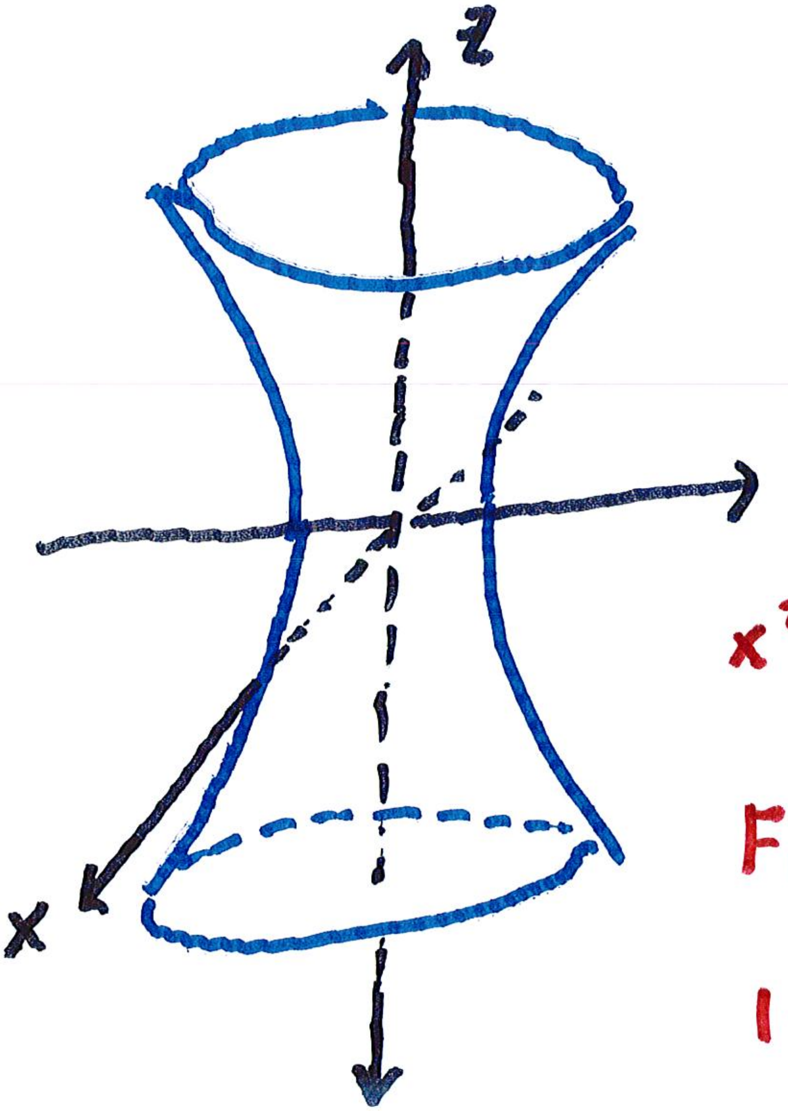
trace is

$$x^2 + y^2 = \left(\sqrt{4+z^2}\right)^2$$

This is a circle of

radius $\sqrt{4+z^2} \to \infty$ as

$z, -z \to \infty$



$x^2 + y^2 = 1 + z^2$

For each z,

$1 + z^2 \geq 1$

This is called a

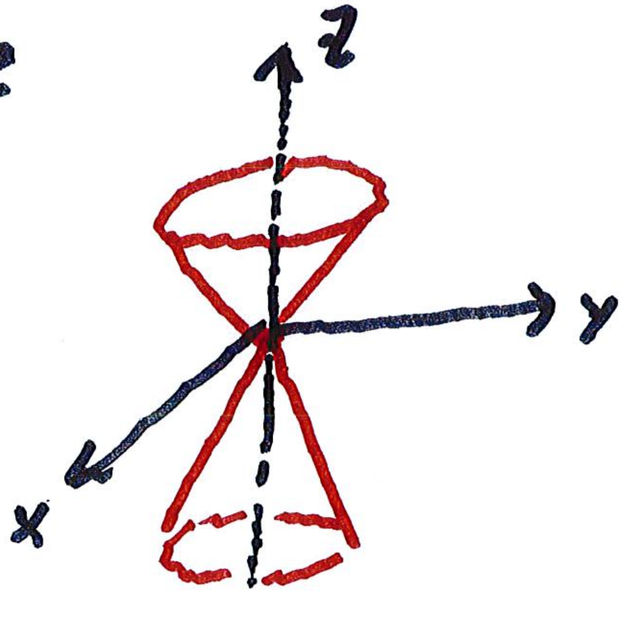
"hyperboloid of 1 sheet"

Ex. Now suppose that

$$x^2 + y^2 - z^2 = 0$$

or $x^2 + y^2 = z^2 = |z|^2$

This is a circle of radius $|z|$. We obtain a <u>cone</u>



Ex. Reduce the equation to a standard form, and sketch it.

$$x^2 - y^2 + z^2 - 2x + 2y + 4z = 0.$$

$$(x^2 - 2x + 1) - (y^2 - 2y + 1)$$

$$+ (z^2 + 4z + 4) = 0 + 1 - 1 + 4 = 4$$

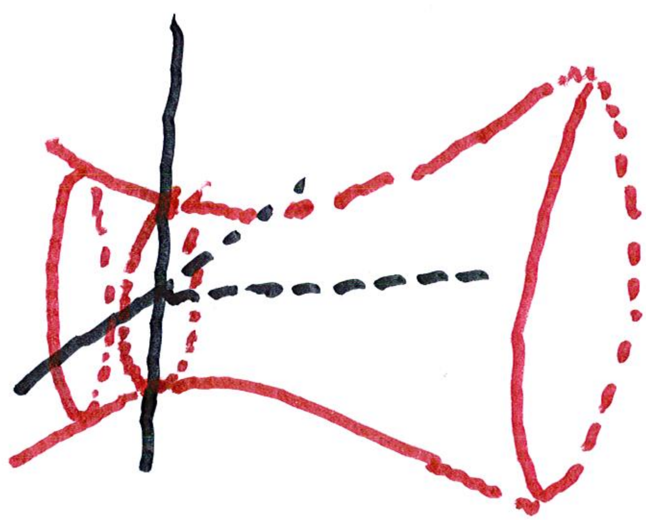
Set $x - 1 = X$, $y - 1 = Y$, $z + 2 = Z$

Equation becomes

$$X^2 - Y^2 + Z^2 = 4$$

$$\therefore \; X^2 + Z^2 = 4 + Y^2$$

For each fixed Y, $4 + Y^2 \geq 4$.



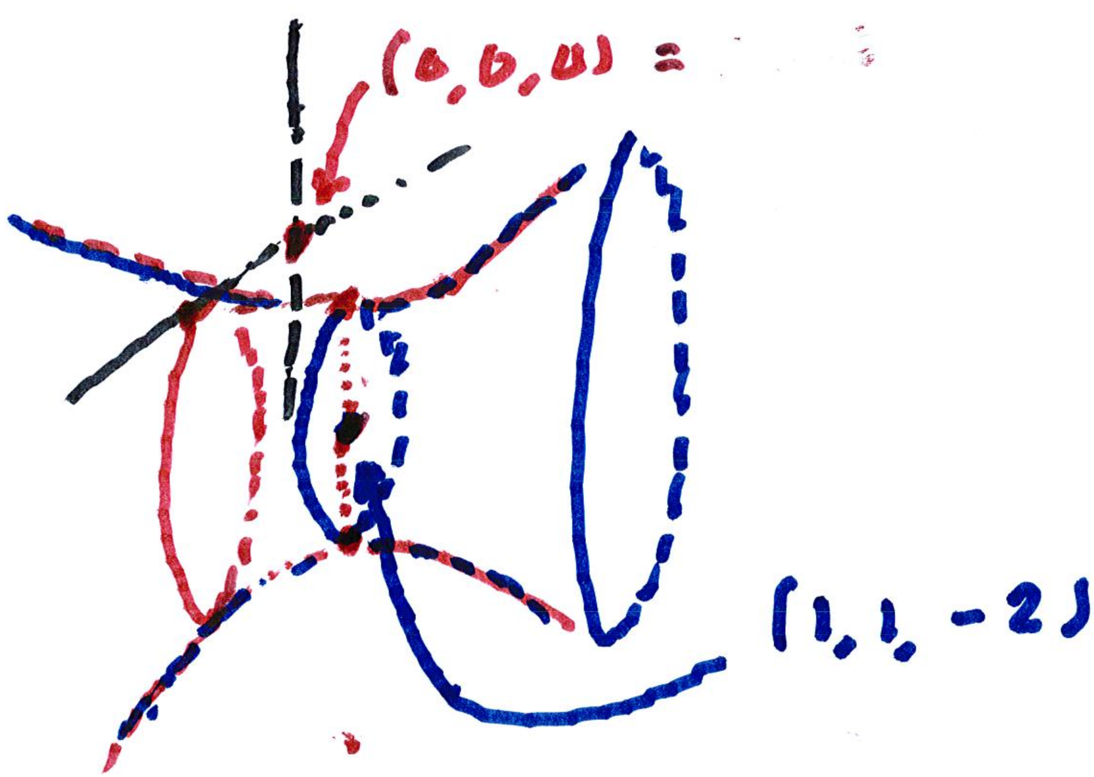
Hyperboloid of 1 sheet.

Now solve for x, y, and z:

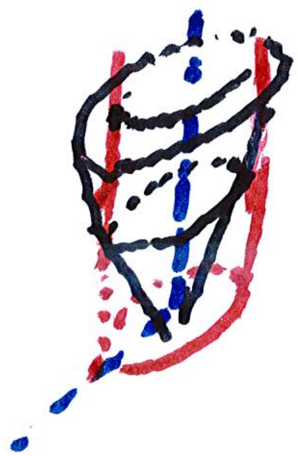
$$x = X + 1, \quad y = Y + 1, \quad z = Z - 2$$

Translate surface so

that the center is (1, 1, -2)



Ex. Sketch the surface bounded by $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 = 1$ for $1 \le z \le 2$



$z^2 = x^2 + y^2$

(cone with axis: z-axis)