Theorem. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D, then $\vec{F}$ is conservative, i.e., there is a function $f(x,y)$ so that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.

Ex. Find f so that

$$\frac{\partial f}{\partial x} = \underbrace{y^2 - 3x^2}_{P} \quad \text{and} \quad \frac{\partial f}{\partial y} = \underbrace{2xy + 2y}_{Q}$$

First check if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 2y \quad ✓$$

Note that P and Q are defined in $\mathbb{R}^2$, which is simply-connected. ✓

Since $P = y^2 - 3x^2$,

$f(x,y) = \int (y^2 - 3x^2)\,dx$ $\quad \frac{\partial f}{\partial x} = y^2 - 3x^2$

$= xy^2 - x^3 + h(y).$

Now differentiate in y:

$$\frac{\partial f}{\partial y} = 2xy + h'(y)$$

But $Q(x,y) = 2xy + 2y$.

$\therefore$ We need $h'(y) = 2y$

$\rightarrow h(y) = y^2 \quad \therefore f(x,y) = xy^2 - x^3 + y^2$

Compute $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F} = (y^2 - 3x^2)\,\vec{i} + (2xy + 2y)\,\vec{j}$$

and C = curve defined by

$$\vec{r}(t) = \left(\frac{2}{t+1} - t^2\right)\vec{i} + \left(t^3 + \frac{1}{t+2}\right)\vec{j}$$

for $0 \le t \le 1$

Since $\vec{F}$ is conservative,

$\int_C \vec{F} \cdot d\vec{r}$ depends only on the

endpoints. $\vec{r}(0) = \left(2, \frac{1}{2}\right)$

and $\vec{r}(1) = \left(0, \frac{4}{3}\right)$

$$f(\vec{r}(0)) = 2 \cdot \frac{1}{4} - 8 + \frac{1}{4}$$

$$= -8 + \frac{3}{4} = -\frac{29}{4}$$

and

$$f(\vec{r}(1)) = 0 - 0 + \left(\frac{4}{3}\right)^2$$

$$\therefore f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= \left(\frac{4}{3}\right)^2 - \left(-\frac{29}{4}\right)$$

16.4 Green's Theorem

Let C be the boundary of a region D. We adopt the convention the curve C has a positive orientation if C has the counterclockwise orientation. Thus if C is parameterized by $\vec{r}(t)$, $a \le t \le b$, then the domain D should be on the left side

Green's Thm. Let C be positively oriented, and suppose also that the boundary of D is parameterized by a single closed curve C.

If P and Q have continuous partial derivatives, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Sometimes you'll see the notation

$\oint_C P\,dx + Q\,dy$ to emphasize that C is parameterized counter-clockwise.

In one variable, the Fund. Thm.

of Calculus states that

$$\underbrace{\int_a^b F'(x)\,dx}_{\text{On the interior of } [a,b]} = \underbrace{F(b) - F(a)}_{\text{On the boundary of } [a,b]}$$

Why is Green's Thm. true?

We assume that D is

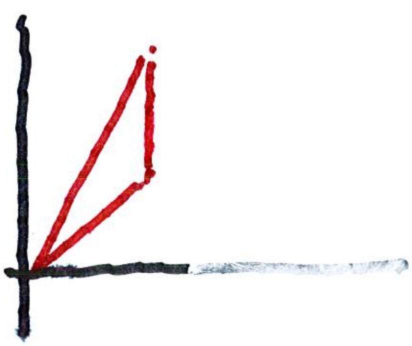
of type I. (Maybe later)

Ex. Use Green's Thm. to

evaluate $\displaystyle\int_C 2xy\,dx + (x^2 - x)\,dy$

where C is the triangle with

vertices at $(0,0)$, $(1,1)$, $(1,3)$



$P(x,y) = 2xy$

$Q(x,y) = x^2 - x$

$\dfrac{\partial Q}{\partial x} = 2x - 1 \qquad \dfrac{\partial P}{\partial y} = 2x$

$$\therefore \int_C P\,dx + Q\,dy$$

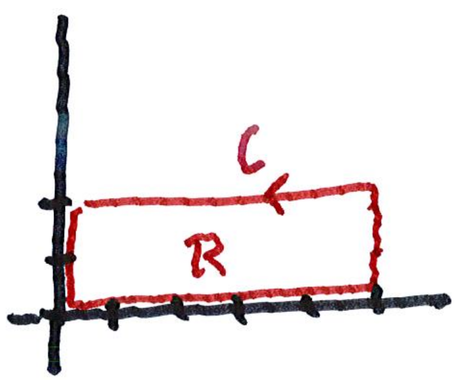
$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D \left((2x-1) - (2x)\right) dA$$

$$= -\iint_D = -A = -(1 \cdot 2 \cdot \tfrac{1}{2}) = -1$$

Ex. Compute $\int_C \cos y \, dx + x^2 \sin y \, dy$,

where C = rectangle with vertices at $(0,0)$, $(5,0)$, $(5,2)$ and $(0,2)$.



$P(x,y) = \cos y$

$Q(x,y) = x^2 \sin y$

$$\int_C = \iint_R (2x \sin y) + \sin y \; dA$$

$$= \int_{\substack{0\\x}}^{5} \int_{\substack{0\\y}}^{2} (2x+1) \sin y \, dy \, dx$$

$$= \int_{0}^{5} (2x+1)(-\cos y) \Big|_{0}^{2}$$

$$= \int_{0}^{5} (2x+1)(-\cos 2 + 1) \, dx$$

$$= (1 - \cos 2)(x^2 + x) \Big|_{0}^{5}$$

$$= \underline{\underline{(1 - \cos 2) \cdot (30)}}$$

Ex. Use Green's Thm. to evaluate

$$\int_C \vec{F} \cdot d\vec{r}, \quad \text{where } \vec{F} = \langle y - \cos y,\ x\sin y \rangle$$

and C = circle

$$(x-3)^2 + (y+4)^2 = 4.$$

cent. = $(3, -4)$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA =$$

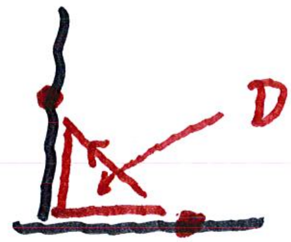
$$= \iint_D \sin y - (1 + \sin y)\, dA$$

$$= -\iint_D 1 \, dA = -\pi \cdot 2^2 = -4\pi$$

Ex. Use Green's Thm. to find the work done by the force

$\vec{F} = x(x+y)\,\vec{i} + xy^2\,\vec{j}$ in moving

a particle from the origin along the x-axis to $(1,0)$, then along the line segment to $(0,1)$ and then back along the y-axis

to the origin 

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (y^2 - x)$$

$$\therefore \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_D y^2 - x \; dA$$

$$= \int_0^1 \int_0^{1-x} (y^2 - x)\, dA$$

$$= \int_0^1 \left(\frac{y^3}{3} - xy\right)\Big|_0^{1-x}$$

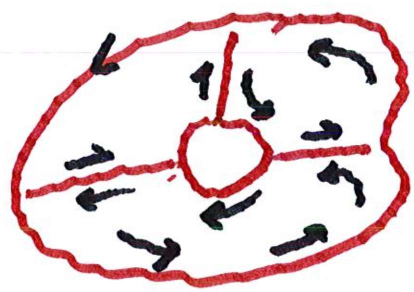
$$\int_0^1 \frac{(1-x)^3}{3} - x(1-x) \, dx$$

$$= \int_0^1 \frac{1}{3} - x + x^2 - \frac{x^3}{3} - x + x^2 \, dx$$

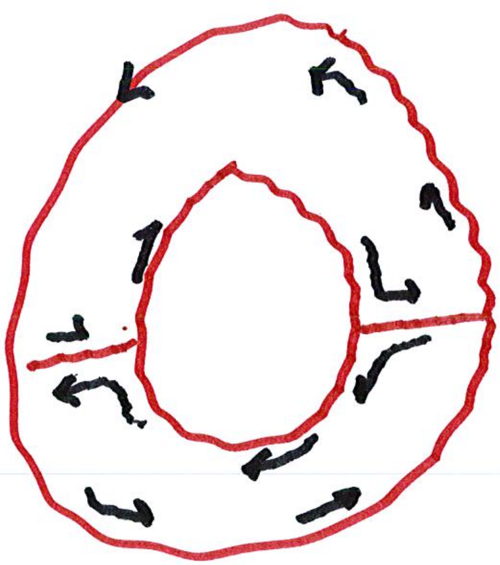
$$= \int_0^1 \frac{1}{3} - 2x + 2x^2 - \frac{x^3}{3} \, dx$$

$$= \frac{1}{3} - 1 + \frac{2}{3} - \frac{1}{12} = -\frac{1}{12}$$

What about more complicated regions



Note that the <u>inside</u> is <u>parameterized clockwise.</u>



We can use Green's Thm.

to compute the area of a

domain D

There are 3 ways :

1. Set $P = 0$, $Q = x$

$\therefore Q_x - P_y = 1 - 0 = 1$

2. Set $P = -y$, $Q = 0$

$\therefore Q_x - P_y = 0 - (-1) = 1$

or

3. $Q = \frac{1}{2}x$, $P = -\frac{1}{2}y$

$\therefore Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

Ex. Find area of ellipse. (Use 3)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\iint_D 1+1\,dA = A$$

Set $x = a\cos t$ $y = b\sin t$

$$A = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos t \cdot b\cos t - b\sin t \cdot (-a\sin t)\,dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\,dt$$

$$= \underline{\underline{\pi ab}}$$

Ex. 10 Let D be the region

between the circles $x^2+y^2=4$

and $x^2+y^2=9$

C = boundary of D

$$\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$$

$$= \iint (3x^2+0)-(0-3y^2)\,dA$$

$$= 3\iint_D x^2+y^2\,dA$$

$$= 3 \int_0^{2\pi} \int_2^3 r^2 \cdot r \, dr \, d\theta$$

$$= 6\pi \left. \frac{r^4}{4} \right|_2^3$$

$$= \frac{3\pi}{2} (81 - 16)$$

Calculate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F} = (e^{-x} + y^2) \vec{i} + (e^{-y} + x^2) \vec{j}$$

C = closed path =

(0,1)

half-disk

(0,-1)

By Green's Thm.

$$\int_C \vec{F} \cdot d\vec{n} = \iint 2x - 2y \quad dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r\cos\theta \cdot r\,dr\,d\theta$$

$$= \frac{2}{3} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{4}{3}$$