16.5 Curl

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$,

where P, Q, and R are all

different. We define curl $\vec{F}$ by

$$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

We can cross product to make

this simpler

We write a formal product as follows:

We write

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

We apply ∇f by

$$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

We can think of ∇ as a vector

with components $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

Then

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j}$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= \operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

Ex. If $\vec{F} = yz\vec{i} - xyz\vec{j} + y^2\vec{k}$,

find curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xyz & y^2 \end{vmatrix}$$

$$= (2y - xz)\vec{i} - (0 - y)\vec{j} + (-yz - z)\vec{k}$$

$$= (2y - xz)\vec{i} + y\vec{j} + (-yz - z)\vec{k}$$

Thm. If $\vec{F} = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$.

then curl (∇f).

In fact

$$\text{curl}(\nabla f) = \nabla \times (\nabla f)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j}$$

$$+\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = \vec{0}$$

$\therefore$ curl $\vec{F} \neq 0$. i.e. curl (grad f) $= \vec{0}$

So, if $\vec{F} = \nabla f$, it must be that $\underline{\underline{\text{curl } \vec{F} = \vec{0}}}$

As an example, if $\vec{F}$ = vector field in the first example, then $\underline{\text{curl } \vec{F} \neq 0}$, $\underline{\text{so } \vec{F} \neq \nabla f.}$

Thm. If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ and curl $\vec{F} = 0$, then there is a function f with $\nabla f = \vec{F}$

Ex. Let $\vec{F} = x^2 z^3 \vec{i} + 2xyz^3 \vec{j} + 3x^2 y z^2 \vec{k}$

(a) Show $\vec{F}$ is conservative, i.e., that curl $\vec{F} = \vec{0}$.

Ex. Show that

$$\vec{F} = y^2 z^3 \vec{i} + 2xyz^3 \vec{j} + (3xy^2z^2 + z)\vec{k}$$

is conservative

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 + z \end{vmatrix}$$

$$= (6xyz^2) - (6xyz^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k}$$

$\therefore \vec{F}$ is conservative.

Now find f so $\nabla f = \vec{F}$.

f must satisfy

1. $\dfrac{\partial f}{\partial x} = \cancel{y^3 z}\ y^2 z^3$

2. $\dfrac{\partial f}{\partial y} = 2xyz^3$

3. $\dfrac{\partial f}{\partial z} = 3xy^2z^2 + -z^2$

Int. 1. $f(x,y,z) = xy^2z^3 + g(y,z)$

$f_y = 2xyz^3 + g_y$

$= 2xyz^3$

Take y - derivation

$\cancel{f_y(x, y, z) =}$

plug into 2.

$$2xyz^3 + g_y = 2xyz^3$$

$$\Longrightarrow g(y,z) = h(z) \quad \text{plug into 3}$$

$$\therefore f(x,y,z) = xy^2z^3 + h(z)$$

$$f_z = 3xy^2z^2 = 3xy^2z^2 + h'(y)$$

$$\therefore h'(y) = -\frac{z^3}{3}$$

$$f(x,y,z) = xy^2z^3 - \frac{z^3}{3}$$

Divergence.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

Then

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Using ∇,

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (P, Q, R)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If $\vec{F} = 2xy\,\vec{i} - xzy^2\,\vec{j} + z^2\,\vec{k}$,

then $\nabla \cdot \vec{F}$

$$= 2y - 2xzy + 2z$$

Thm. If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

is a vector field, then

div curl $\vec{F} = 0$

Ex. Show $\vec{F} = xyz\,\vec{i} + yz\,\vec{j} - xyz\,\vec{k}$

is not = curl of a vector field G.

$\text{div}\,\vec{F} = yz + z - xy \neq 0$

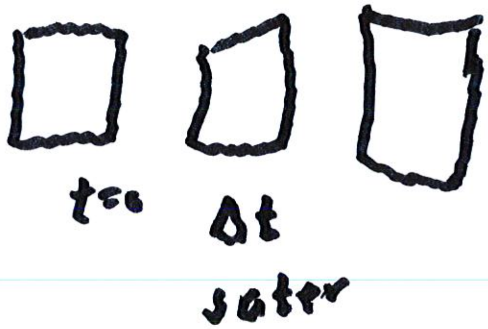
$\therefore\ \vec{F} \neq \text{curl}\ G$

(For if $F = \text{curl}\ G$, then

$\text{div}\,\vec{F} = \text{div}\,\text{curl}\ G = 0$)

divergence $\vec{F}$ measures the

rate of fluid to expand its volume



div $\vec{F} > 0$ means fluid is expanding.

Another important differential expression is

$\nabla^2 = \nabla \cdot \nabla$. When applied to f,

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \doteq 0.$$

We can also form

$$\nabla^2 \vec{F} = \nabla^2 (P\vec{i} + Q\vec{j} + R\vec{k})$$

$$= \nabla^2 P\,\vec{i} + \nabla^2 Q\,\vec{j} + \nabla^2 R\,\vec{k}$$

Vector forms of Green's Thm.

Given $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$,

we can view take curl of $\vec{F}$

(assuming coef. of $\vec{k} = 0$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

$$\Rightarrow \text{curl } \vec{F} \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \cdot \vec{k}$$

$$= \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}$$

We can rewrite Green's Thm as

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cdot \vec{k}\, dA$$

Suppose $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$

$$\Rightarrow \vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$

One can show that the outward pointing normal is

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\,\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}$$

Check $\vec{n}(t)$ is $\perp$ to $T(t)$

$$\therefore \int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}\cdot\vec{n}\;|\vec{r}'(t)|$$

$$= \int_a^b \frac{P(x,y)\,y'(t) - Q(x,y)\,x'(t)}{|\vec{r}'(t)|} \;\; d \;\; |\vec{r}'(t)|\,dt$$

$$= \int_a^b P(x(t), y(t))\, y'(t) - Q(x(t), y(t))\, x'(t)\, dt$$

$$= \int_C P\,dy - Q\,dx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

This is the Divergence Thm