Curl.

Suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

has coefficients that are C_1.

We define

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j}$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Ex. If $\vec{F} = (x^2 - y)\vec{i} + 4z\vec{j} + x^2\vec{k}$

Find curl $\vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix}$$

$$= (0 - 4)\vec{i} - (2x - 0)\vec{j} + (0 + 1)\vec{k}$$

$$= -4\vec{i} - 2x\vec{j} + \vec{k}$$

Using Vector notation (∇)

$\text{curl}\,\vec{F} = \nabla \times \vec{F}$ and if

f is a scalar vection,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

Hence, del f = grad f,

Important Identity.

curl grad f = 0

or $\nabla \times \nabla f = 0$.

16.6 Parametric Surfaces

\# 38. p. 1098

Maxwell's Equations. Let $\vec{E}(x_1, x_2, x_3, t)$ and $\vec{H}(x_1, x_2, x_3, t)$ denote the electrical and magnetic field at (x_1, x_2, x_3, t)

Maxwell showed that $\vec{E}$ and $\vec{H}$ satisfy

$$\operatorname{div} \vec{E} = \vec{0} \qquad \operatorname{div} \vec{H} = 0$$

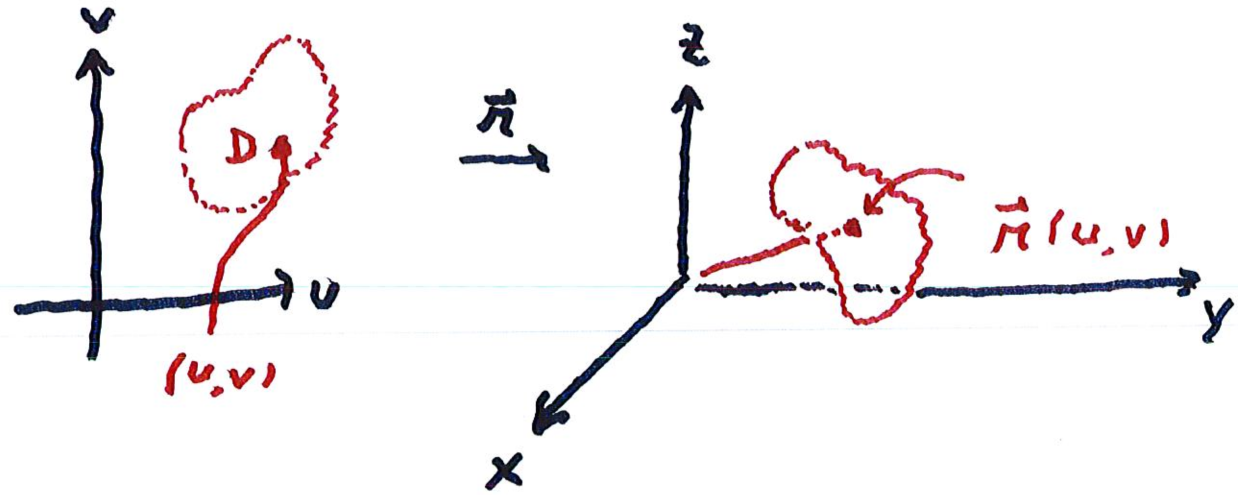
$$\operatorname{curl} \vec{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad \operatorname{curl} \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

He showed that light is an electro-magnetic wave with c = speed of light.

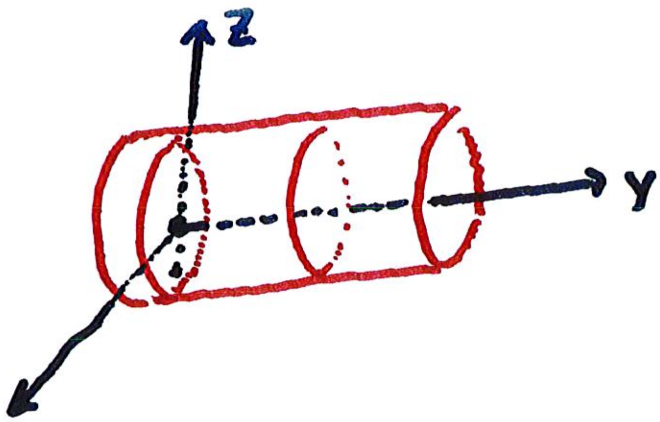
16.6 We can describe a surface by

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

As u and v change they describe a surface in $\mathbb{R}^3$.



Ex. A cylinder of radius 3 :



$$x = 3\cos u \,, \quad z = 3\sin u \,, \quad y = v$$

or:

$$\vec{r}(u,v) = 3\cos u\,\vec{i} + v\vec{j} + 3\sin u\,\vec{k}$$

Spherical Coord:

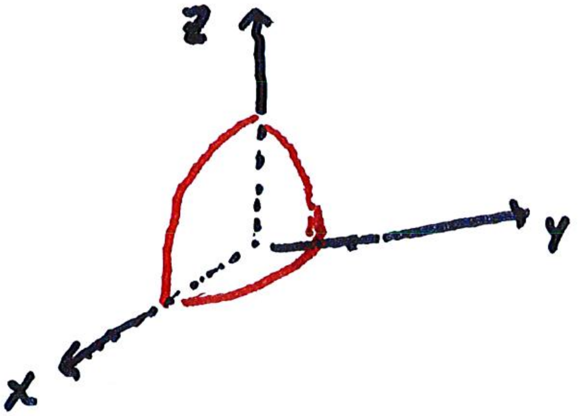
$$x(\theta,\phi) = 4\cos\theta\sin\phi$$

$$y(\theta,\phi) = 4\sin\theta\sin\phi$$

$$z(\theta,\phi) = 4\cos\phi$$

$0 < \theta < \frac{\pi}{2}$

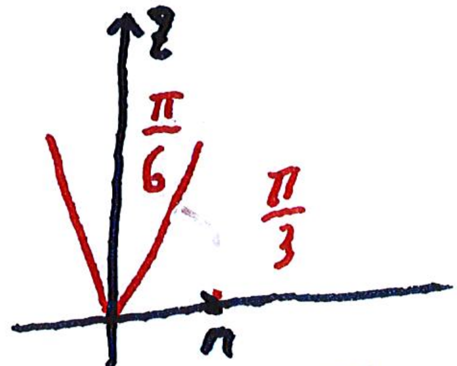
$0 < \phi < \frac{\pi}{2}$



A sphere of radius 4 in first octant

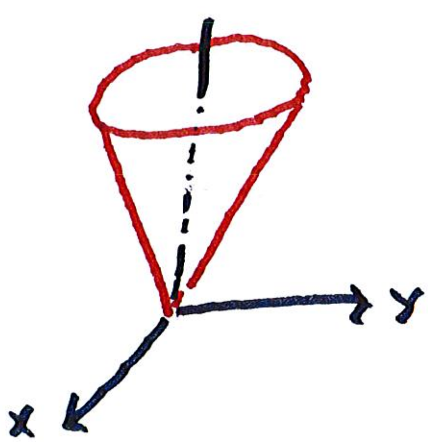
Ex. Give a parameterization of the top half of the cone with angle $\frac{\pi}{6}$ from the central axis

First when $x = 0$



$z = \sqrt{3}\, r$

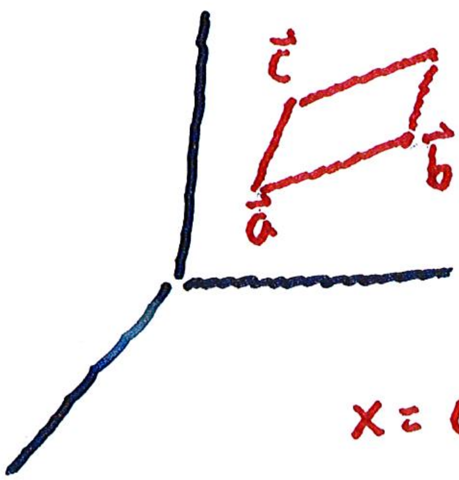
Now with $x \neq 0$, $z = \sqrt{3}\sqrt{x^2 + y^2}$



$\vec{a}$, $\vec{b}$, and $\vec{c}$:

$$(u,v) \to \vec{a} + u(\vec{b} - \vec{a}) + v(\vec{c} - \vec{a})$$

for $0 \leq u \leq 1$ and $0 \leq v \leq 1$



$$\vec{r}(u,v) = u(\vec{b} - \vec{a}) + v(\vec{c} - a)$$

$$x = a_1 + u(b_1 - a_1) + v(c_1 - a_1)$$

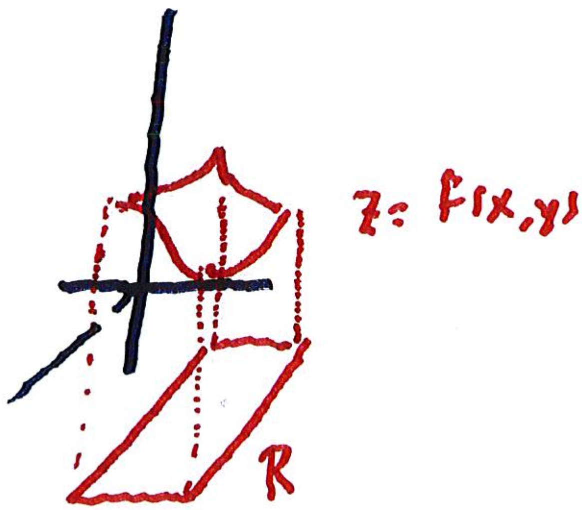
$$y = a_2 + u(b_2 - a_2) + v(c_2 - a_2)$$

$$z = a_3 + u(b_3 - a_3) + v(c_3 - a_3)$$

Write a parameterization

of the graph of $f(x,y)$ for

$(x,y) \in D$



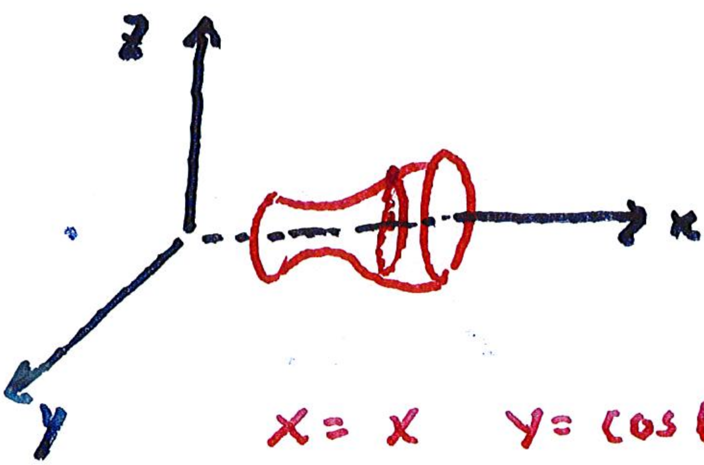
S = graph of
f over R

$$(u,v) \in R \xrightarrow{\vec{r}} (u, v, f(u,v))$$

Surfaces of Revoution

Revolve $y = f(x)$, $a \le x \le b$

about the x-axis



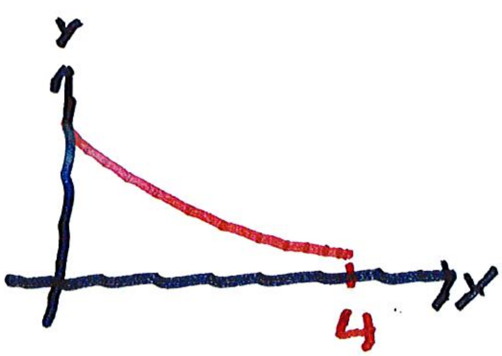
$x = x \quad y = \cos\theta \, f(x) \quad z = \sin\theta \, f(x)$

when $\theta = 0$, we get the usual

curve $y = f(x)$.

when $\theta =$, we get $y = f(x)$

Ex. Find a parameterization of the surface obtained by rotating the curve $y = e^{-x}$ for $0 \le x \le 4$ about the x-axis



$x = x$

$y = e^{-x} \cos\theta$

$z = e^{-x} \sin\theta$

$0 \le \theta \le 2\pi$

$\therefore (x, \theta) \rightarrow \quad = \left(x,\ \cos\theta\, f(x),\ \sin\theta\, f(x)\right)$

for $0 \le x \le 4$

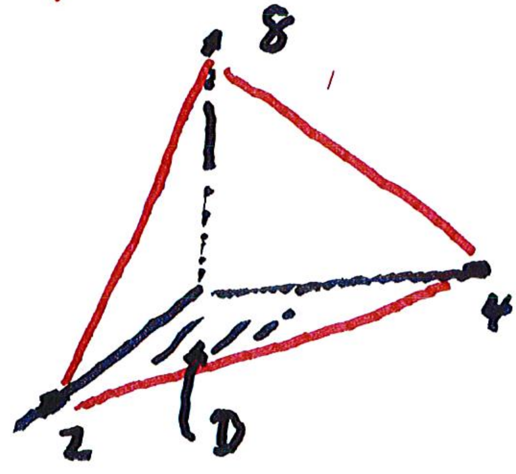
and $0 \le \theta \le 2\pi$

Find a pamameterization

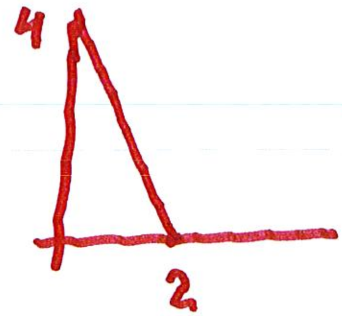
of the triangle in the first

octant generated by

$$4x + 2y + z = 8$$



Let D = base in the plane $z=0$



$4x+2y=8$

or $2x+y=4$

$0<z<8-4x-2y$

$$\boxed{\begin{array}{c} 0<y<4-2x \\ x>0,\ y>0 \end{array}}$$

D

or $0 \le v \le 4-2u$

$u \ge 0, \quad v \ge u$

$z = 8 - 4\ \ - 2$

We study how $\vec{r}(u,v)$ when we only let u change

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\,\vec{k}$$

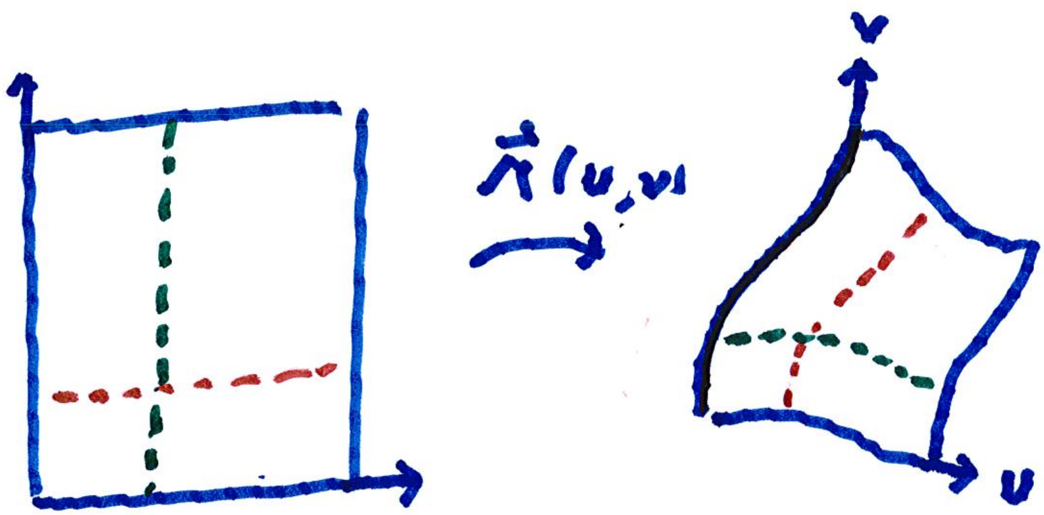
and then only let v change:

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\,\vec{k}$$

First, we want to allow

u to change when $v = v_0$ is

fixed.



This gives a curve

$$u \to \vec{r}(u, v_0) \approx \vec{r}(u_0, v_0) + \frac{\partial \vec{r}}{\partial u}(u_0, v_0)(u - u_0)$$

Similarly, thegreen curve

is approximated by

$$v \to \vec{r}(u_0, v) \approx \vec{r}(u_0, v_0)$$

$$+ \frac{\partial \vec{r}}{\partial v}(u_0, v_0)(v - v_0) .$$

If we subdivide the intervals

by $u_0 < u_1, \dots, u_{m-1}, < u_m$

and $v_0 < v_1, \dots < v_{m-1} < v_n$

If we choose various values of u_0 and v_0, we obtain a family of grid curves.

The horizont and vertical sides of the red square are mapped to $\frac{\partial \vec{r}}{\partial u}(u_0, v_0)\,\Delta x$ and $\frac{\partial \vec{r}}{\partial v}(u_0, v_0)\,\Delta y$

These sides generate a parallelogram with

$$\text{area} \approx \left| \frac{\partial \vec{r}}{\partial u_0}(u_0, v_0) \times \frac{\partial \vec{r}}{\partial v_0}(u_0, v_0) \right| \cdot \Delta u \, \Delta v .$$