16.7 Surface Integrals

Suppose S is defined by

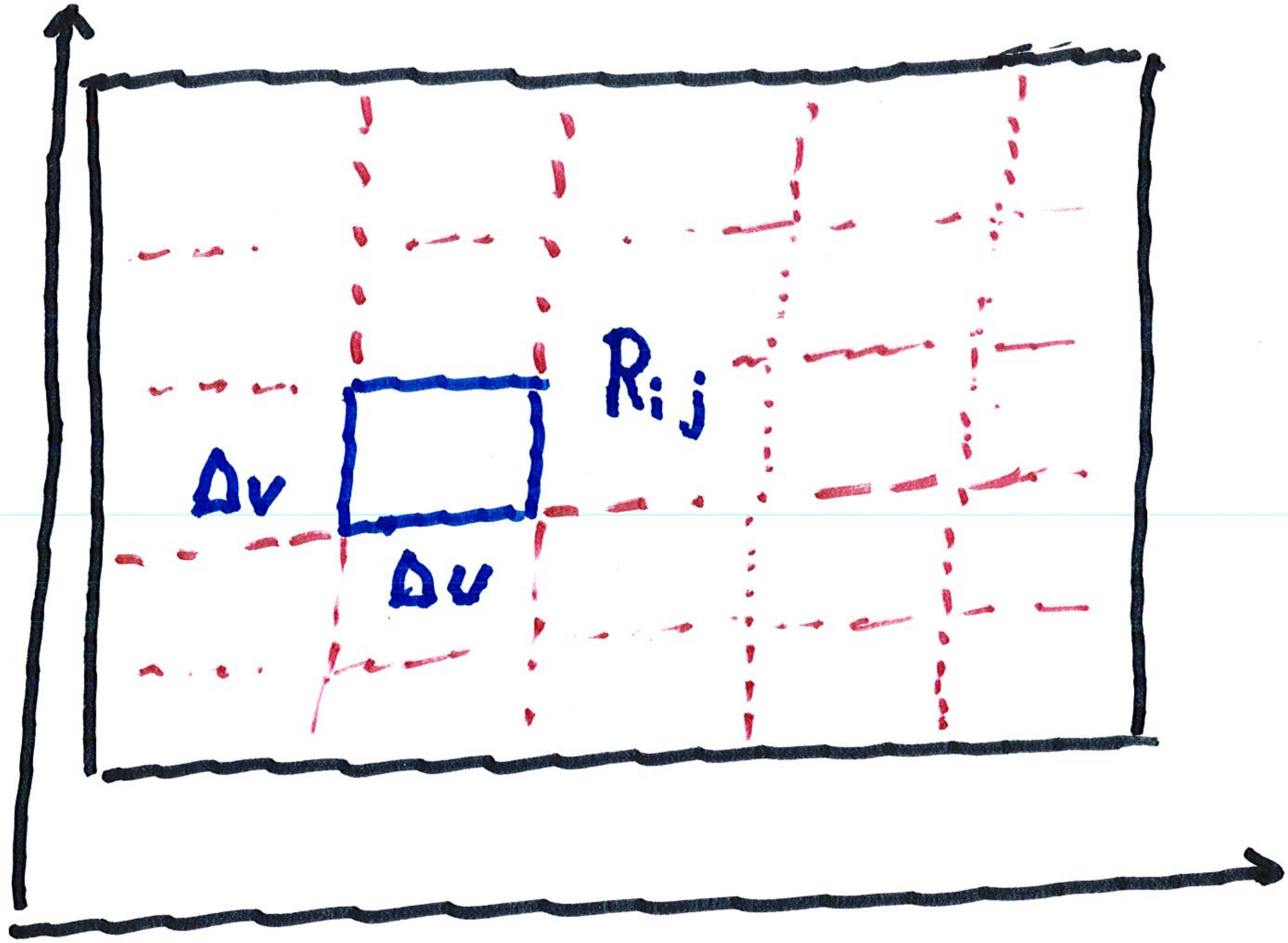
$\vec{r}(u,v)$

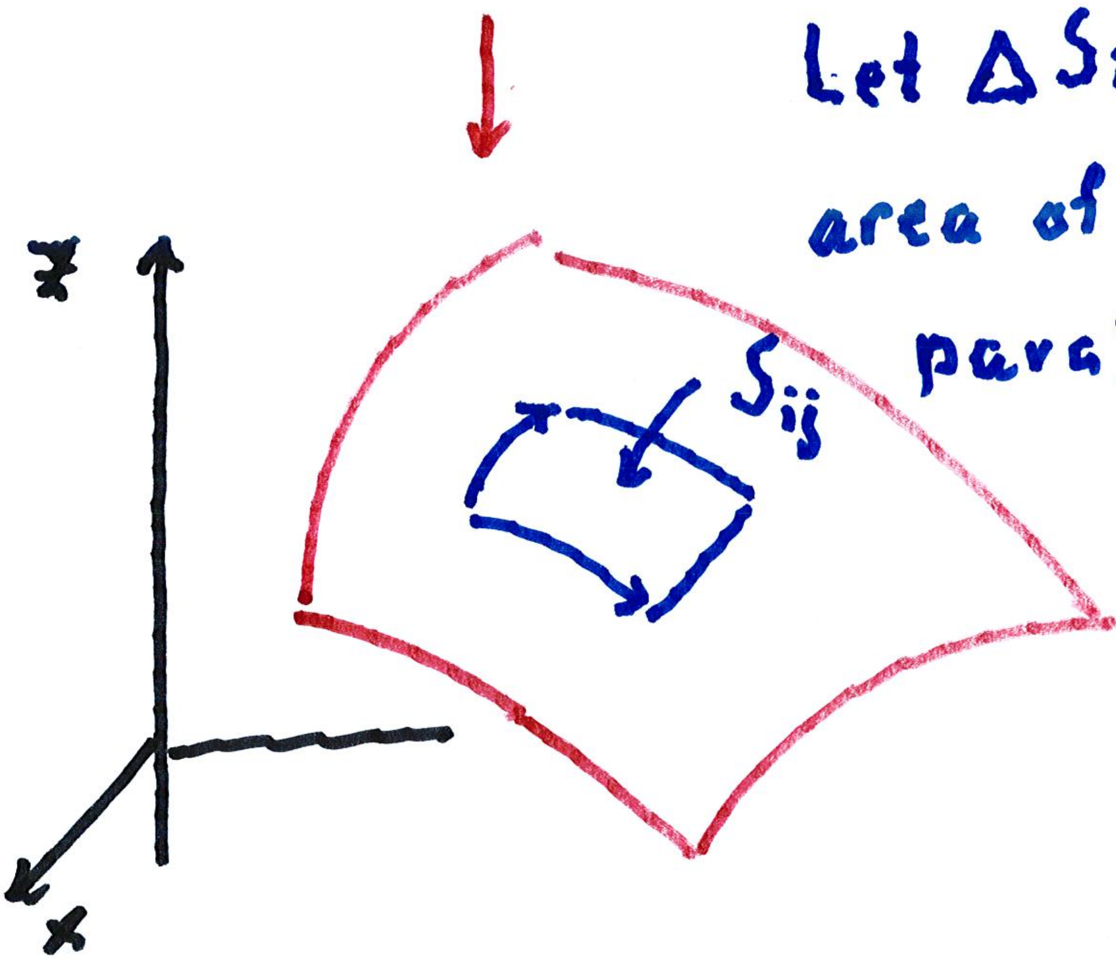
$= x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{j}$

for $(u,v) \in D$.

Recall S can be described

as a union of parallelograms:





Let ΔS_{ij} = area of small parallelogram

This formula is similar to the 1-dimensional situation,

where

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

Now we study Graphs of functions

Any surface S with equation $z = g(x,y)$ can be viewed as a parametric surface

with

$x = x$, $y = y$ and $z = g(x, y)$

so $\vec{r}_x = \vec{i} + \left(\dfrac{\partial g}{\partial x}\right)\vec{k}$

and $\vec{r}_y = \vec{j} + \left(\dfrac{\partial g}{\partial x}\right)\vec{k}$.

Hence

$$\vec{r}_x \times \vec{r}_y = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}$$

Hence,

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right) + 1}\,.$$

Again, let $F(x, y, z)$ be any continuous function defined near S. We obtain

$$\iint_S f(x, y, z)$$

$$= \iint_D f(x, y, z)\, dS\,.$$

$$= \iint_D f(x, y, z) \sqrt{1 + (g_x)^2 + (g_y)^2}\; dA$$

Surface Integrals cont.d

Let S be the portion of the unit sphere in the first octant. Calculate the surface integral (Use sherical coordinates)

$$\iint_S xy\, dS$$

Since $\rho = 1$, the sphere is defined by

$$\boxed{x = \sin\phi\cos\theta,\quad y = \sin\phi\sin\theta \quad \text{and} \quad z = \cos\phi.}$$

$$\iint_S xy \, dS$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3\phi \, \sin\theta \cos\theta \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \cdot \int_0^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta$$

$$\downarrow$$

$$\frac{1}{2}\sin^2\theta \Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$\downarrow$$

$$\int_0^{\frac{\pi}{2}} (1 - \cos^2\phi) \sin\phi \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi + \frac{\cos^3\phi}{3} \, d\phi$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Answer is

$$\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

Note that the area of the parallelogram patch on S is

$$\left|\vec{r}_u \times \vec{r}_v\right| \Delta u \, \Delta v$$

Let $f(x, y, z)$ be a continuous function defined near S

We chouse a point P^*_{ij} in the parallelogram patch S_{ij}

If we let m and $n \to \infty$,

then we see that

$$A(S) = \iint |\vec{r}_u \times \vec{r}_v| \, dA$$

Given a function $f(x,y)$

on a ~~function~~ domain D,

we define $S = \{(x, y, f(x,y))$; where $(x,y) \in D\}$

We define a parameterization

of ~~the~~ S by

To study S_{ij}, we first fix v_0 and let v vary. This gives a smooth function of v, $\vec{\pi}(v)$, that can be approximated by

$v \longrightarrow \vec{\pi}(v) =$.

$$\vec{\pi}(v) = \vec{\pi}(v_0) + \vec{\pi}(v)\underbrace{(v - v_0)}_{\Delta v}$$

Similarly, by fixing u_0,

we obtain a function $\vec{r}(v)$

that can be approximated

by

$$\vec{r}(v) = \underset{\vec{r}(v_0)}{\vec{r}(v)}\,\underbrace{(v-v_0)}_{\Delta v}$$

This gives 2 short segments

$\vec{r}_u \Delta u$ and $\vec{r}_v \Delta v$,

which gives a small

parallelogram

of area $|\vec{r}_u \times \vec{r}_v|\,\Delta u\,\Delta v$

If we set P_{ij} to be any

point in S_{ij}, we have

shown that the surface

S_{ij} has surface area

$$|\vec{r}_u(P_{ij}) \times \vec{r}_v(P_{ij})|\,\Delta u\,\Delta v.$$

As always we add up these

quantities, and let $m, n \to \infty$.

We obtain the formula

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

and

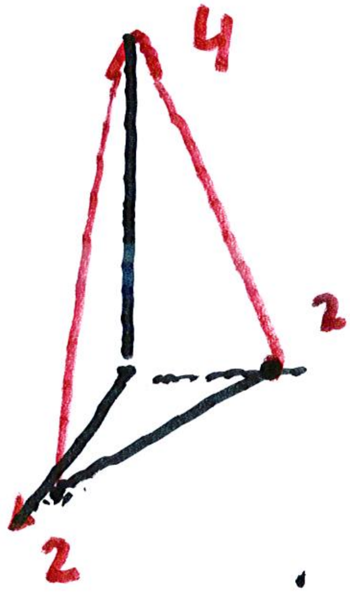
$$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k} .$$

#10 Find $\iint_S xz\,dS$ if

S is the part of the plane

$2x + 2y + z = 4$ that lies in

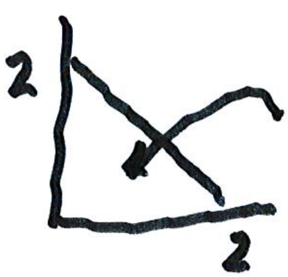
the first octant.



To find the basis,

set $z = 0$.

$\longrightarrow 2x + 2y = 4$



$x + y = 2$

Note that S = graph of

$f(x,y) = 4 - 2x - 2y$

$\therefore \rightarrow \frac{\partial f}{\partial x} = -2 \qquad \frac{\partial f}{\partial y} = -2$

$\rightarrow$ correction factor of surface

is $\sqrt{1+4+4} = 3$

Thus $\iint_S xz \, dS = \iint_D x \cdot (4-2x-2y) \cdot 3 \, dy \, dx$

$$= \int_0^2 \int_0^{4-2x} x\,(\underbrace{4-2x-2y}_{z}) \cdot 3 \, dy \, dx$$

$$= 3 \int_0^2 \int_0^{4-2x} 4x - 2x^2 - 2xy \quad dy\, dx$$

$$= 3 \int_0^2 \int_0^{4-2x} (4x - 2x^2) y - 2xy$$

$$= 3 \int_0^2 \int_0^{4-2x} (4x - 2x^2 - 2x) y$$

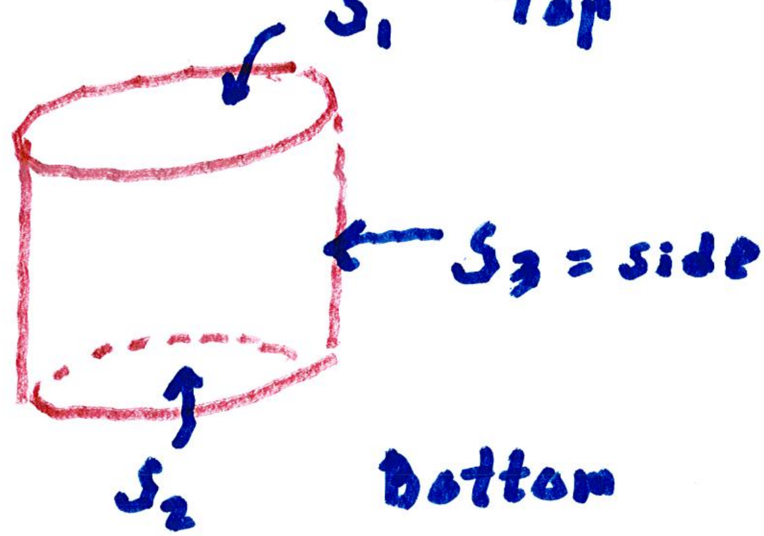
$$= 3 \int_0^2 \int_0^{4-2x} (2x - 2x^2) \left. \frac{y^2}{2} \right|_0^{4-2x}$$

$$= 3 \int_0^0 (2x - 2x^2) \frac{(4-2x)^2}{2} \; dx$$

A surface S is orientable if there is a smoothly varying vector field $\vec{n}$ (usually a unit vector field at all points of the set S.). For the surface $z = g(x,y)$, we can select the upward pointing vector field.

Sometimes, the boundary of a region in $\mathbb{R}^3$ has several pieces

Ex. A cylindrical can height h and width r.



A box has 6 parts.