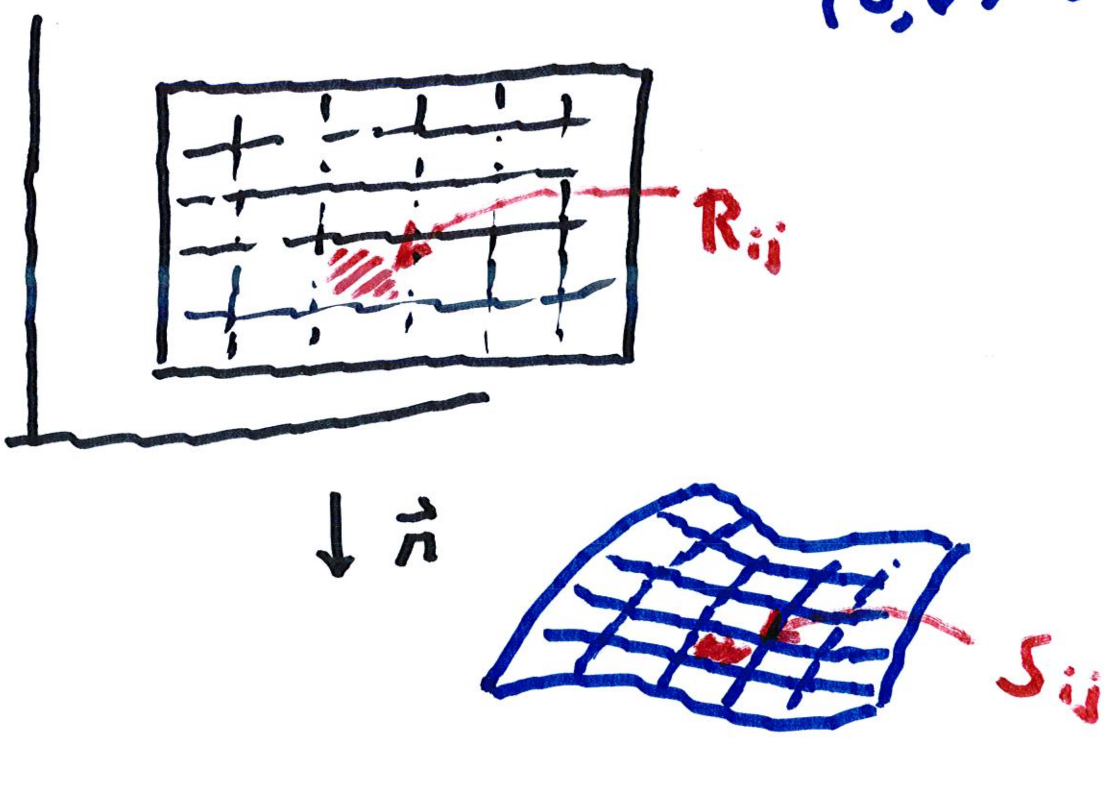
16.7 Surface Integrals.

Suppose that a surface S has a vector equation

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{j}$$

$$(u,v) \in D.$$



We divide R into subrectangles of length Δu and Δv. Then S is divided into patches S_{ij}. We form the Riemann sums (f is defined near S)

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^{*}) \Delta S_{ij}.$$

If we take the limit as $m, n \to \infty$, we get $\iint_S f(x,y,z)\, dS$.

S_{ij} is like a parallelogram

with sides $\vec{r}_u \Delta u$ and $\vec{r}_v \Delta v$.

The area of S_{ij} is

$\approx |\vec{r}_u \Delta u \times \vec{r}_v \Delta v|$. Hence

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v,$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

and $$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

One can show that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,\underbrace{|\mathbf{r}_u \times \mathbf{r}_v|\,dA}\qquad (1)$$

Ex. Compute $\iint_S x^2\,dS$, where

S is the unit sphere.

We use spherical coordinates.

$$x = \sin\phi \cos\theta$$

$$y = \sin\phi \sin\theta$$

$$z = \cos\phi$$

where

$0 \le \phi \le \pi$,

$0 \le \theta \le 2\pi$,

that is,

$$\vec{r}(u,v) = \sin\phi \cos\theta \, \vec{i} + \sin\phi \sin\theta \, \vec{j} + \cos\phi \, \vec{k} .$$

In 16.6 we showed that

$$|r_\phi \times r_\theta| = \sin\phi .$$

Hence

$$\iint_S x^2 \, dS = \iint_D (\sin\phi \cos\theta)^2 \, |r_\phi \times r_\theta| \, dA$$

Note $|r_\phi \times r_\theta| = \sin\phi$

$$= \int_0^{2\pi} \int_0^{\pi} \sin^2 \cos^2\theta \sin\phi$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\theta \, d\phi$$

$$= \int_0^{2\pi} \frac{1}{2}\left(1 + \cos 2\theta\right) d\theta$$

$$\cdot \int_0^{\pi} \sin O - \cos^2\phi \, d\phi$$

$$= \frac{4\pi}{3}$$

Graphs.

A surface defined by

$z = g(x,y)$ is defined by a parametric equation

$$x = x, \quad y = y, \quad z = f(x,y),$$

so we have

$$\vec{n}_x = \vec{i} + \left(\frac{\partial g}{\partial x}\right)\vec{k} \qquad \vec{n}_y = \vec{j} + \left(\frac{\partial g}{\partial y}\right)\vec{k}$$

$$\therefore \ \hat{n}_x \times \hat{n}_y = -\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}.$$

and $\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$

Hence, (2) becomes

$$\iint_S f(x,y,z)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\;dA.$$

If $\vec{v}$ is a vector field, we write

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S F\cdot\vec{n}\;dS.$$

(called the flux of F acro

Recall $\vec{F}\cdot\vec{n}$ = component of

$\vec{F}$ in the $\vec{n}$ direction.

Since $\vec{r}_u \times \vec{r}_v$ is $\perp$ to S,

we write

$$\iint_S \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS$$

$$= \iint \left[\vec{F}(\vec{r}(u,v)) \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \right.$$

$$dA$$

Thus, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

Thus the integral $\iint_S \vec{F} \cdot d\vec{S}$

is converted into an integral

over the parameter domain

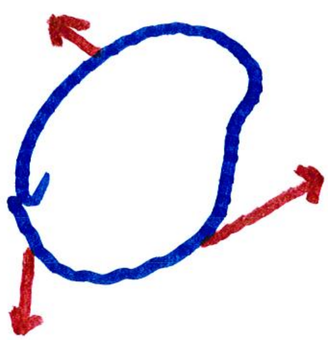
by including the factor $\vec{r}_u \times \vec{r}_v$.

Stokes' Thm.

Suppose that C is the boundary of an oriented surface S. Then

$$\int_C \vec{F} \cdot d\vec{x} = \iint \nabla \times F \cdot \vec{n} \, dS .$$

$\nabla \times F = \text{curl } \vec{F}$



positive circulation

To state Stokes' Thm., we need to calculate curl $\vec{F}$

If $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$,

we can write

$$\text{curl } \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

It's easier to write

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Ex. Let C be the

intersection of the paraboloid

$z = x^2 + y^2$ and the plane $z = y$,

and give C its counterclockwise
orientation
as viewed from the pan

positive z-axis.

Evaluate

$$\int_C xy\,dx + x^2\,dy + z^2\,dz.$$

Let $\vec{F}(x, y, z) = xy\,\vec{i} + x^2\vec{j} + z^2\vec{k}$

and let Σ be the part of

the plane $z = y$ that lies inside

the paraboloid. By (Stokes) we

can ~~evaluate~~ evaluate the

given line integral by

evaluating $\displaystyle\iint_{\Sigma} (\text{curl } \vec{F}) \cdot n \, dS.$

Note that if (x, y, z) is on C,

then $x^2 + y^2 = y$, and this is

an equation of the

circular cylindrical having

equation $r = \sin\theta$ in

cylinder coordinates.

Therefore if R is the

region ~~is in the x-y~~

in the xy plane bounded by the ~~normal~~ circle $r = \sin\theta$, then Σ is the graph of $z = y$ on R. When we orient Σ by the normal, directed upward, the induced orientation on C is counterclockwise.

Since

$$\operatorname{curl} \vec{F}(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & z^2 \end{vmatrix}$$

$$= x\vec{k}$$ and

$$f(x,y) = y.$$

so we conclude that

$$\int_C xy\,dx + x^2\,dy + z^2\,dz$$

$$= \iint (\text{curl } \vec{F}) \cdot \vec{n}\,dS$$

$$= \iint_{\Sigma} \cancel{\tfrac{1}{3}}\left[-0(0) - 0(1) + x\right] dA$$

$$= \iint_R x\,dA = \int_0^{\pi}\int_0^{\sin\theta} (r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3} \cos\theta \Big|_0^{\sin\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi} \sin^3\theta \cos\theta \, d\theta$$

$$= \frac{1}{12} \sin^4\theta \Big|_0^{\pi} = 0$$