Divergence Thm.

Let E be a simple solid region, and let S be the boundary surface of E, with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an on open region that contains E

Then

$$\iint_S \vec{F} \cdot dS = \iiint_E \operatorname{div} \vec{F} \, dV.$$

The Divergence Thm

states that the flux of $\vec{F}$

across the boundary surface

of E is the triple integral

of the divergence of $\vec{F}$ over E.

Ex. Let S = unit sphere

$x^2 + y^2 + z^2 = 1$ and let

$\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$.

Calculate the flux

integral $\int_S \vec{F} \cdot dS$

Sol'n. Note that

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x)$$

$$= 1.$$

S = boundary of unit ball.

Hence, the Divergence Thm gives the flux as

$$\iint \vec{F} \cdot dS = \iiint_B \text{div } F$$

$$= \iiint_B 1 \, dV = V(B) = \frac{4\pi}{3}$$

Ex. Let S be the surface of the box enclosed by the planes

$x=0,\ x=a,\ y=0,\ y=b,\ z=0,\ z=c,$

where a, b, c are all positive

and let

$$\vec{F}(x,y,z) = x^2yz\,\vec{i} + xy^2z\,\vec{j} + xyz^2\,\vec{k}.$$

Calculate $\displaystyle\iint_S \vec{F}\cdot d\vec{S}$.

Sol'n.

Note that

$$\operatorname{div} \vec{F} = 2xyz + 2xyz + 2xyz$$

$$= 6xyz.$$

~~Stoke's~~ Div. Thm. Thm implies that

$$\iint_S F \cdot d\vec{S} = \iiint 6xyz \, dV$$

$$\int_0^a x \, dx \int_0^b y \, dy \int_0^c z \, dz$$

$$= 6 \frac{a^2}{2} \quad = \frac{3 a^2 b^2 c^2}{4}$$

Ex. Let S be the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

Let $\vec{F} = 3xy^2\,\vec{i} + xe^z\,\vec{j} + z^3\,\vec{k}$.

Calculate $\displaystyle\iint_S \vec{F} \cdot d\vec{S}$.

Sol'n.

Note that

$$\operatorname{div} \vec{F} = 3y^2 + 3z^2$$

If we integrate this over the solid cylinder, we get

$$3 \cdot 3 \cdot \int_0^{2\pi} r^2 \cdot r \, dr \, d\theta$$

$$= 18\pi \cdot \frac{1}{4} = \frac{9\pi}{2}.$$

However, we have not included S_t (for the top) and S_b (for the bottom).

If we integrate this over the solid cylinder, we get

$$\int_{-1}^{2} \iint_{D} 3(x^2+y^2)\, dz\, dA$$

$$= 3 \int_{-1}^{1} \int_{0}^{2\pi} r^3 \, dr\, d\theta$$

$$= 3 \cdot 2\pi \cdot 3$$

Ex. Let S = the sphere about the ~~org~~ origin of radius 2, and let

$$\vec{F}(x,y,z) = (x^3+y^3)\vec{i} + (y^3+z^3)\vec{j} + (z^3+x^3)\vec{k}$$

Calculate $\iint \vec{F}\cdot d\vec{S}$.

Clearly,

$$\operatorname{div}\vec{F} = 3x^2 + 3y^2 + 3z^2.$$

By the Div. Thm.,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_B 3(x^2+y^2+z^2)\,dA$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 3\rho^2 \sin\phi \, \rho^2 \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \cdot \int_0^{2} 3\rho^4 \, d\rho \, d\phi \, d\theta$$

sin φ

$$= \frac{3}{5} \cdot 2\pi \cdot \int_0^{\pi} \sin\phi \, d\phi$$

$$= \frac{12\pi}{5}$$

Calculating the Surface for Flux.

The flux of a three-dimensional vector field across an oriented surface S in the direction of $\vec{n}$ is given by

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{5} \cdot 2 \cdot 2\pi = \frac{12\pi}{5}$$

Ex. Find the flux of

$$\vec{F} = yz\,\vec{j} + z^2\vec{k}$$

outward through S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$ by the planes $x=0$ and $x=1$.

Sol'n. The outward normal field on S may be calculated

$$= \frac{dA}{z}$$

Ex. Let Σ be the cube

described by

$0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1.$

and let $\vec{F}(x, y, z)$ be

$$\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z^2\vec{k}.$$

Calculate $\iint_{\Sigma} \vec{F} \cdot d\vec{S}$.

Solution.

Note that $\operatorname{div} F = 1+1+2z$

$= 2(1+z).$

Let D be the solid region

in the above cube.

We obtain

$$\iint_{\Sigma} \vec{F}\cdot d\vec{S} = \iiint_{D} 2(1+z)\, dV$$

$$\underset{x\ y\ z}{D} \quad \downarrow$$

$$\longrightarrow \int_0^1 \int_0^1$$

$x \quad y$

We can drop the absolute value bars because $z \geq 0$ on S

The value of $\vec{F} \cdot \vec{n}$ on the surface is given by the formula

$$\vec{F} \cdot \vec{n} = (yz\,\vec{j} + z^2\vec{k}) \cdot (y\vec{j} + z\vec{k})$$

$$= y^2 z + z^3 = z(y^2 + z^2)$$

$$= z.$$

Therefore, the flux of $\vec{F}$

outward through S is

$$\iint_S \vec{F} \cdot \vec{n} = \iint (z)\left(\frac{1}{z}\, dA\right)$$

$$= \iint_R dA = \text{area}(R_{xy})$$

$$= 3.$$