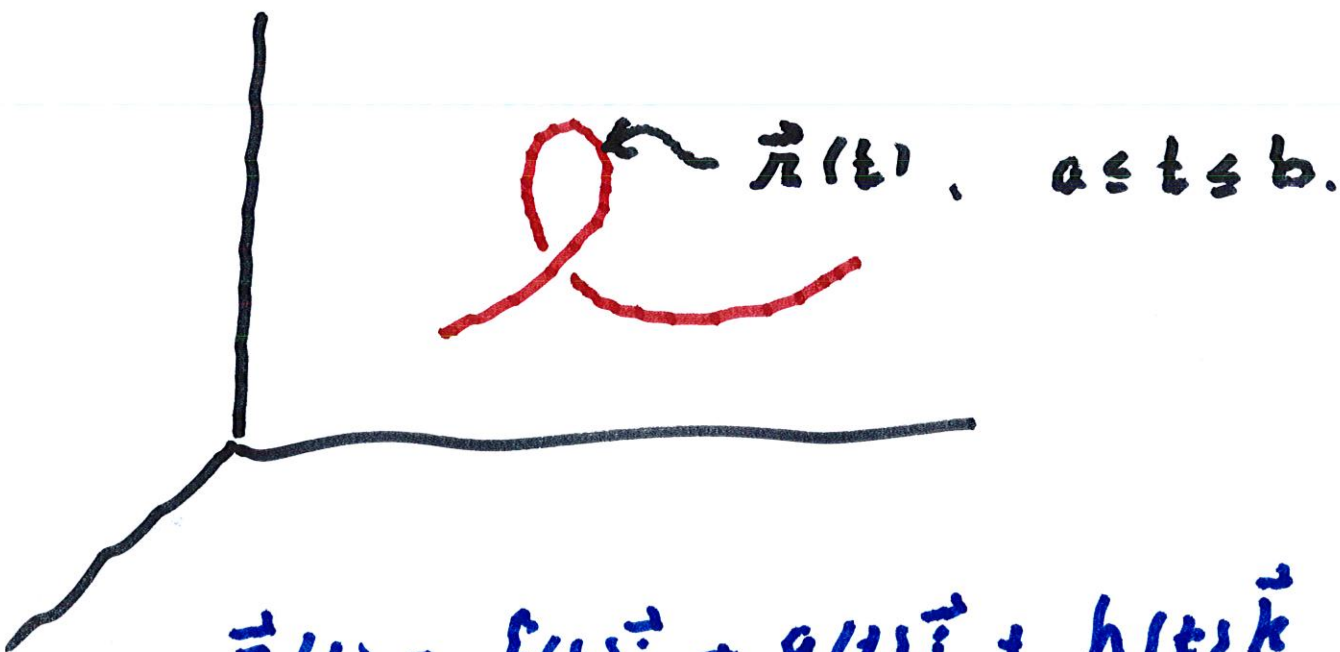
13.3 Arclength and Curvature



$$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

$$a = t_0 < t_1 \dots < t_j < t_{j+1} \dots < t_N = b$$

where $\Delta t = \dfrac{b-a}{N}$

During a Δt-time interval,

$$r(t_{j+1}) - r(t_j) \approx \vec{r}\,'(t_j)\,\Delta t$$

so the distance traveled is

$$\left| \vec{r}(t_{j+1}) - \vec{r}(t_j) \right| \approx |\vec{r}\,'(t_j)|\,\Delta t.$$

Adding all distances:

$$\sum_{j=0}^{N-1} \left| \vec{r}(t_{j+1}) - \vec{r}(t_j) \right| = \sum_{j=0}^{N-1} |\vec{r}\,'(t_j)|\,\Delta t$$

As $N \to \infty$, the total distance is

$$L = \text{Length} = \int_a^b |\vec{r}\,'(t)|\, dt$$

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$,

then $|\vec{r}\,'(t)| = \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2 + \{h'(t)\}^2}$

Ex. Compute the length L of the path $\langle \cos t, \sin t, \ln\{\cos t\} \rangle$ for $0 \le t \le \frac{\pi}{4}$

$$\vec{r}'(t) = \left\langle -\sin t,\ \cos t,\ \frac{-\sin t}{\cos t} \right\rangle$$

$$\therefore L = \int_0^{\pi/4} \sqrt{\sin^2 t + \cos^2 t + \frac{\sin^2 t}{\cos^2 t}}$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 t}\ \ dt$$

$$= \int_0^{\frac{\pi}{4}} \sec t\, dt = \ln|\sec t + \tan t\Big|_0^{\frac{\pi}{4}}$$

$$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right|$$

$$- \ln\left|\sec 0 + \tan 0\right|$$

$$= \ln\left(\sqrt{2} + 1\right)$$

Ex. Find length of the path

$$\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} + \frac{t^3}{3}\,\vec{k}$$

for $0 \le t \le 1$.

$$\vec{r}'(t) = 2\vec{i} + 2t\vec{j} + t^2\vec{k}$$

$$L = \int_0^1 \sqrt{4 + 4t^2 + t^4}\, dt$$

$$= \int_0^1 \sqrt{(2+t^2)^2}\, dt$$

$$= \int_0^1 2 + t^2\, dt = 2t + \frac{t^3}{3}\Big|_0^1$$

$$= 2 + \frac{1}{3} = \frac{7}{3}$$

Ex. Express the length of the path $\vec{r}(t) = \langle t^2, t^3, t^4 \rangle$ as an integral. for $0 \le t \le 2$

$$\vec{r}\,'(t) = \langle 2t, 3t^2, 4t^3 \rangle$$

$$\therefore L = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\, dt$$

This cannot be expressed in terms of standard functions

$$= \int_0^2 \sqrt{1 + \frac{9t^2}{4} + 4t^4}\ (2t)\,dt$$

$u = t^2 \quad du = 2t\,dt$

$$= \int_0^4 \sqrt{1 + \frac{9u}{4} + 4u^2}\ du$$

This can be computed by the methods in 162.

Ex. Let C be the curves of intersection of the parabolic cylinder $x^2 = 2y$ and $3z = xy$. from $(0,0,0)$ to $(6, 18, 36)$.

First, let $x = t$. Then $y = \frac{x^2}{2}$, so $y = \frac{t^2}{2}$. Solving for z, we get $z = \frac{xy}{3}$ or $z = \frac{t \cdot t^2}{3 \cdot 2}$

or $z = \frac{t^3}{6}$.

Since $X = t$ goes from 0 to 6.

$$\therefore L = \int_0^6 \sqrt{1^2 + t^2 + \frac{t^4}{4}}\, dt$$

$$= \int_0^6 \sqrt{\frac{4 + 4t^2 + t^4}{4}}\, dt$$

$$= \int_0^6 \frac{1}{2} \sqrt{(2 + t^2)^2}\, dt$$

$$= \int_0^6 \frac{1}{2} (2 + t^2)\, dt$$

$$= \int_0^6 1 + \frac{t^2}{2} \, dt$$

$$= t + \frac{t^3}{6} \Big|_0^6 = 6 + 36 = 42$$

Ex. Reparametrize the path

$$t \longrightarrow \vec{r}(t) = \langle 2t-3,\ 4t+2,\ 4t+5 \rangle$$

by arclength.

First we compute the

distance function

$s(t)$ of the path

$\vec{r}'(t) = \langle 2, 4, 4 \rangle$

$$\therefore \quad s(t) = \int_0^t \sqrt{36}\, du = 6t$$

Hence $s = 6t \rightarrow t = \frac{s}{6}$

$\therefore$ We define $\vec{R}(s)$ by

$\vec{R}(s) = \vec{r}(t(s))$

$$\vec{R}(s) = \left\langle 2\left(\frac{s}{6}\right)-3,\ 4\left(\frac{s}{6}\right)+2,\ 4\left(\frac{s}{6}\right)+5 \right\rangle$$

$$= \left\langle \frac{s}{3}-3,\ \frac{2s}{3}+2,\ \frac{2s}{3}+5 \right\rangle$$

$$|\vec{R}'(s)| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= 1.$$

$\vec{R}(s)$ is a parameterization by arclength: Compute $s(t)$ and solve for t, i.e. Find $t(s)$. $\vec{R}(s) = r(t(s))$

Suppose

$$s(t) = \int_0^t |\vec{r}\,'(u)|\, du$$

This is the distance of the path from $\vec{r}(0)$ to $\vec{r}(t)$

Recall $\dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \vec{T}(t)$

Suppose $\vec{R}(s)$ $(|R'(s)| = 1)$ is parameterized by arclengh. We define the curvature K by

$$K = \left| \frac{d\vec{T}}{ds} \right|.$$

To avoid solving for t.

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

Hence:

$$\kappa = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Ex. Find curvature of circle of radius a:

Let $x = a\cos t$, $y = a\sin t$

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$\vec{r}'(t) = \langle -a\sin t, a\cos t \rangle$

$|\vec{r}'(t)| = a$.

$\therefore \vec{T}(t) = \langle -\sin t, \ \cos t, \rangle$

$\vec{T}'(t) = \langle -\cos t, \ -\sin t \rangle$

$|\vec{r}'(t)| = a$

$$\therefore \ \kappa = \frac{\left|\frac{dT}{dt}\right|}{|\vec{r}'(t)|} = \frac{1}{a}$$

←.... normalizes the speed.

Another formula for

the curvature κ is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} .$$

$\left(\begin{array}{l} \text{When} \\ \vec{r}(t) \in \mathbb{R}^3 \end{array} \right)$

Since $|\vec{T}(t)| = 1$

$\rightarrow \vec{T}(t) \cdot \vec{T}(t) = 1$

$\rightarrow \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t) = 0$

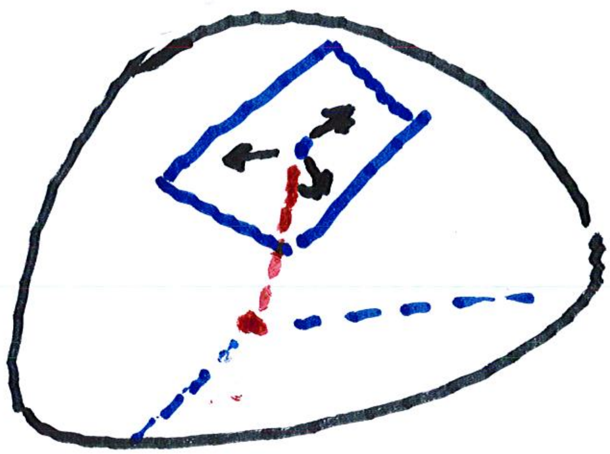
we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$

$\therefore \vec{T}'(t)$ is $\perp$ to $\vec{T}(t)$

We define the unit normal vector

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|T'(t)|}$$

gives the direction $\vec{T}$ is curving in.



$T'(t)$ always moves in a direction that is perpendicular to $T(t)$.

Ex. Let $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$

Find κ and $\vec{N}$

$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

Hence $\vec{T}(t) = \dfrac{-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}}{\sqrt{2}}$

$\therefore \vec{T}\,'(t) = \dfrac{-\cos t\,\vec{i} - \sin t\,\vec{j}}{\sqrt{2}}$

$$\therefore \vec{N}(t) = \frac{\vec{T}'(t)}{|T'(t)|}$$

$$= -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$r'(t)$ = motion vector

$\vec{T}(t)$ = direction of motion.

$\vec{N}(t)$ = direction the particle is curving in.

K = rate of change of $T(s)$

$$= \left| \frac{d\vec{T}}{dt} \Big/ |\vec{r}'(t)| \right|$$

The above curve is on a cylinder of radius 1,

like a winding staircase.

As t changes by 2π, the height (in z variable) increases by 2π

Ex. Find the curvature of

$\vec{r}(t) = \langle t, t^2, t^3 \rangle$

at the point $(1, 1, 1)$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$\therefore \vec{T}(t) = \langle 0, 2, 6t \rangle$

$\rightarrow \vec{T}\,'(t) = \langle 0, 0, 6 \rangle$

Since $K = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$,

$$K(t) = \frac{|\langle 0, 0, 6 \rangle|}{|\langle 1, 2t, 3t^2 \rangle|}$$

$$= \frac{6}{\sqrt{1^2 + 4t^2 + 9t^4}}$$ = curvature at $\langle t, t^2, t^3 \rangle$

The point $\langle 1, 1, 1 \rangle$ corresponds to $t = 1$. Hence the curvature K at $\langle 1, 1, 1 \rangle$ is

$$\frac{6}{\sqrt{1+4+9}} = \underline{\underline{\frac{6}{\sqrt{14}}}} .$$

Ex. Find an expression for

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\left(\vec{v}(t)\times\vec{w}(t)\right)\right]$$

By Leibnitz' Rule

$$= \vec{u}\,'(t)\cdot\left(\vec{v}(t)\times\vec{w}(t)\right)$$

$$+ \vec{u}(t)\cdot\left(\vec{v}\,'(t)\times\vec{w}(t)\right)$$

$$+ \vec{u}(t)\cdot\left(\vec{v}(t)\times\vec{w}\,'(t)\right)$$