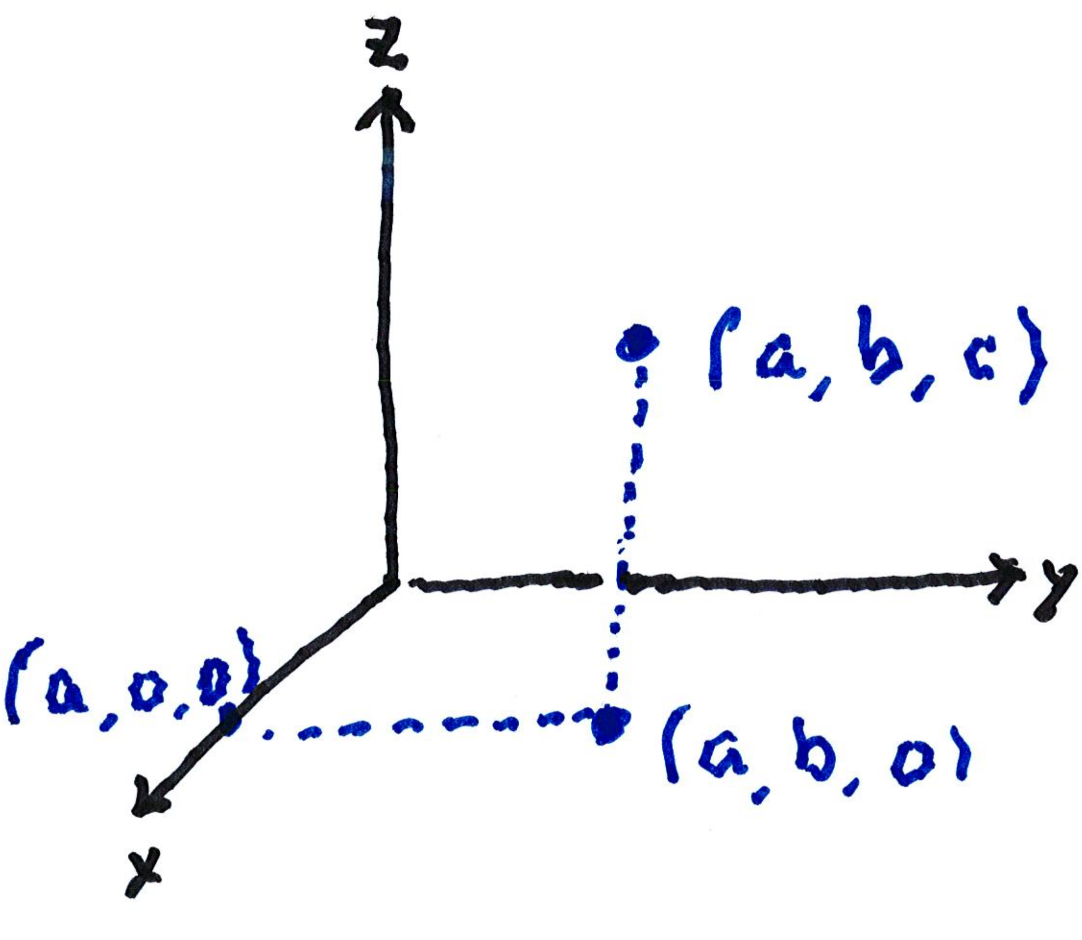
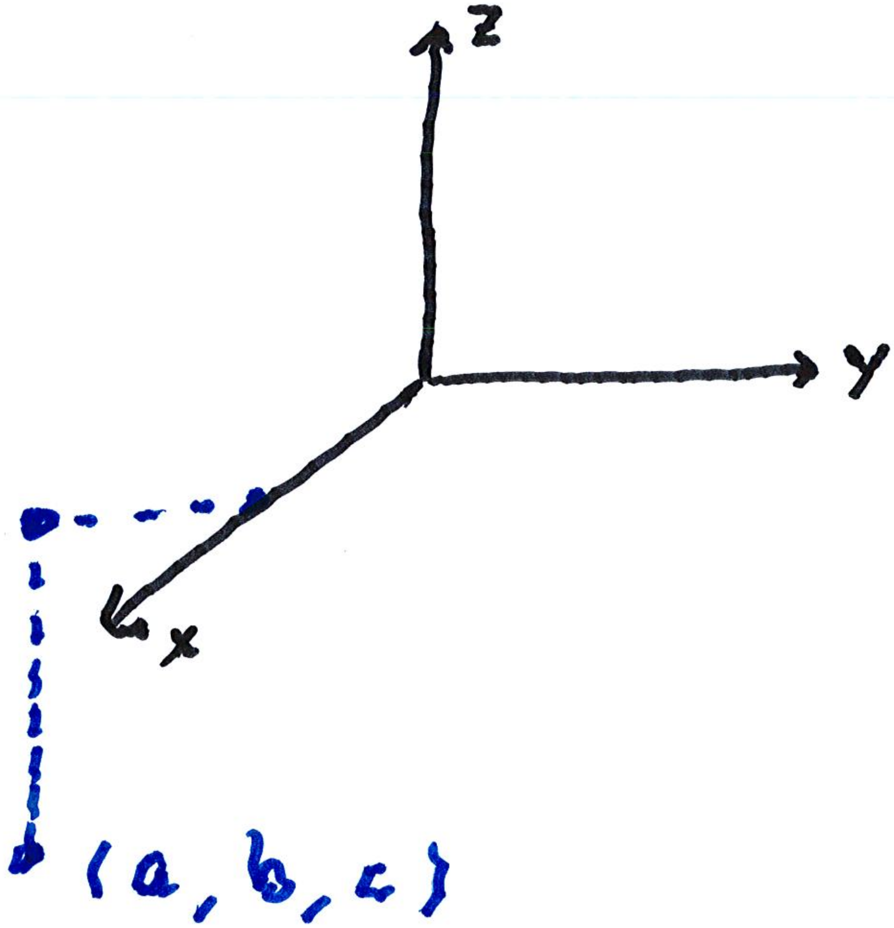
12.1 — 12.4 Review

12.1 3-dimensional coordinates

Each point in 3-dim. space can be written as $P(a, b, c)$



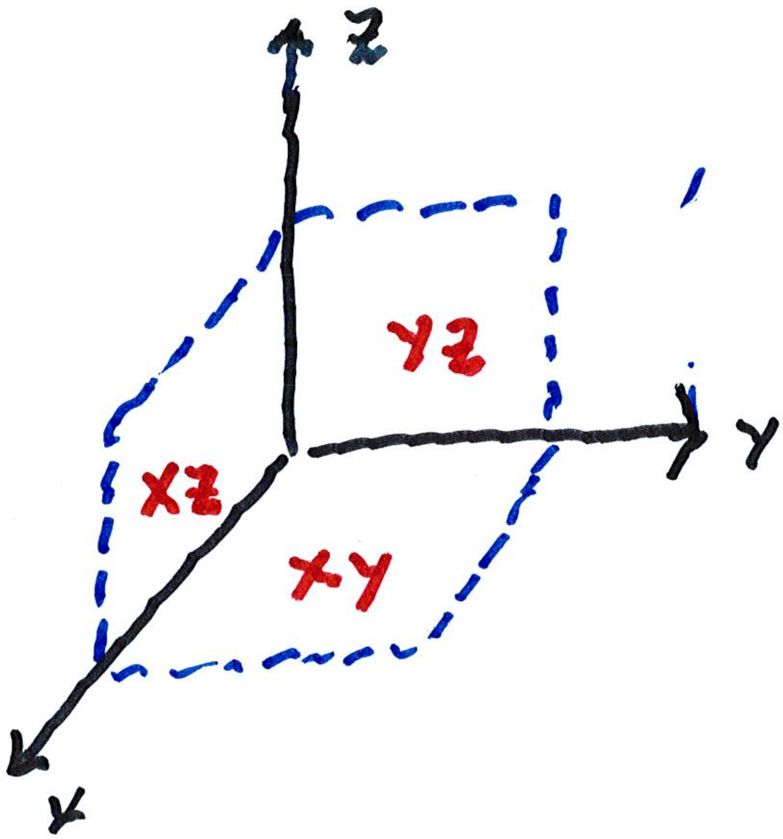
Suppose $a > 0$ and $b, c < 0$



Note that the xy-plane is $z = 0$

The xz-plane is $y = 0$

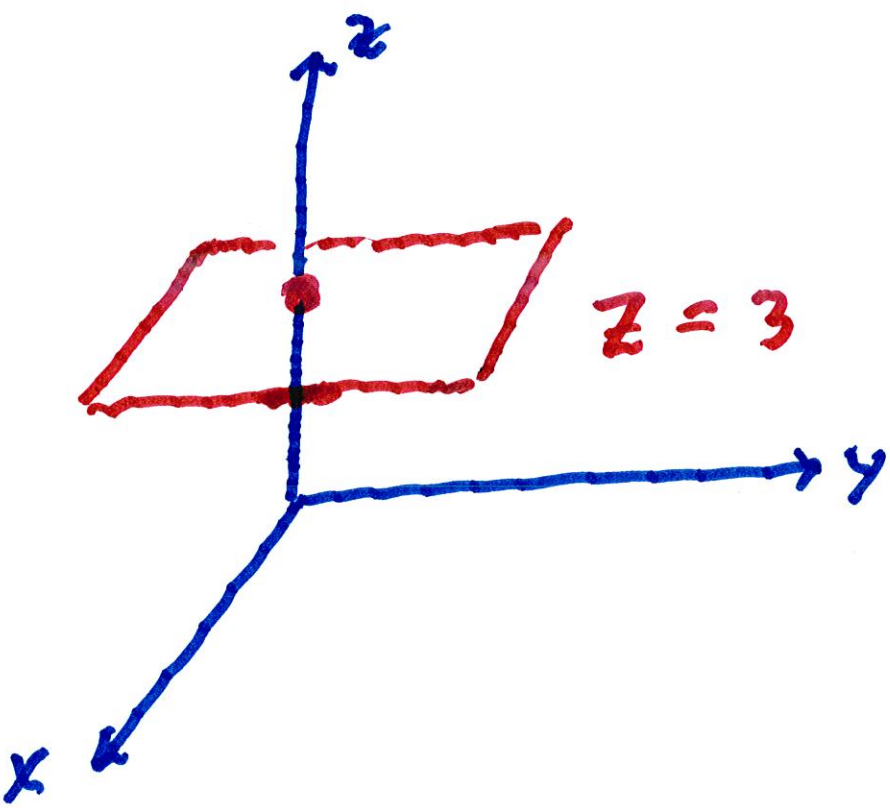
The yz-plane is $x = 0$



The equation $z = 3$

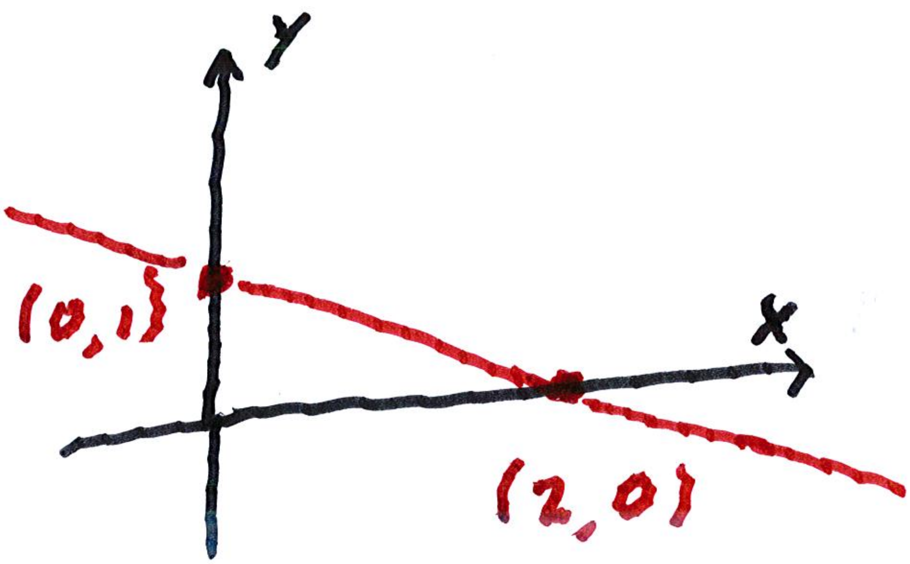
defines a flat plane

through $(0, 0. 3)$

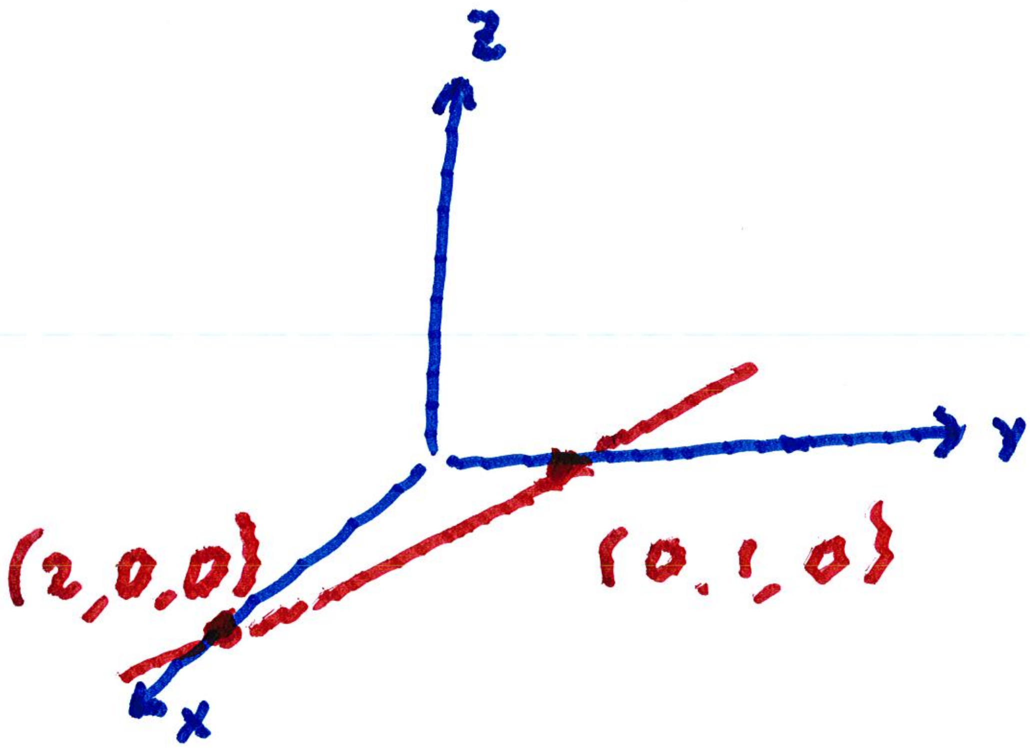


Ex. Sketch $x + 2y = 2$

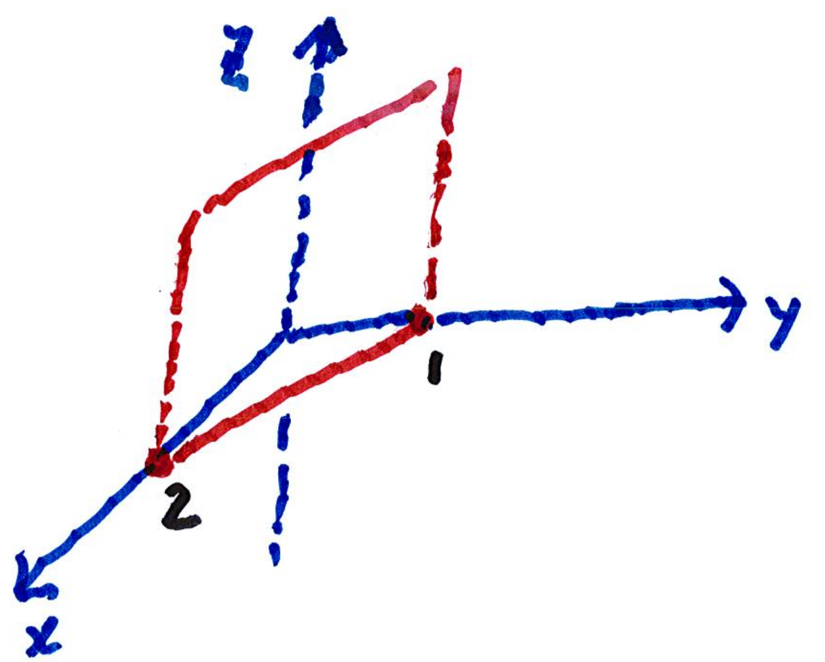
(ind. of z)



Put this line in xy-plane:

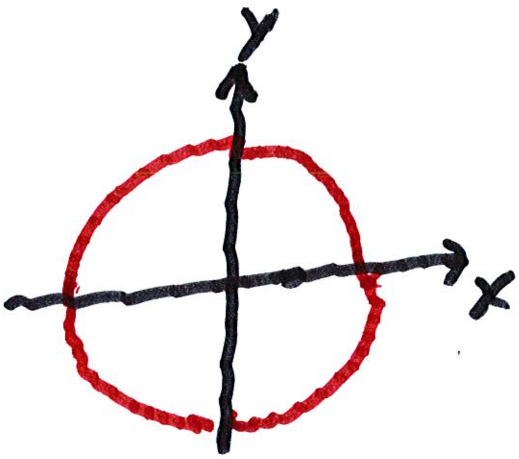


Slide the line up and down



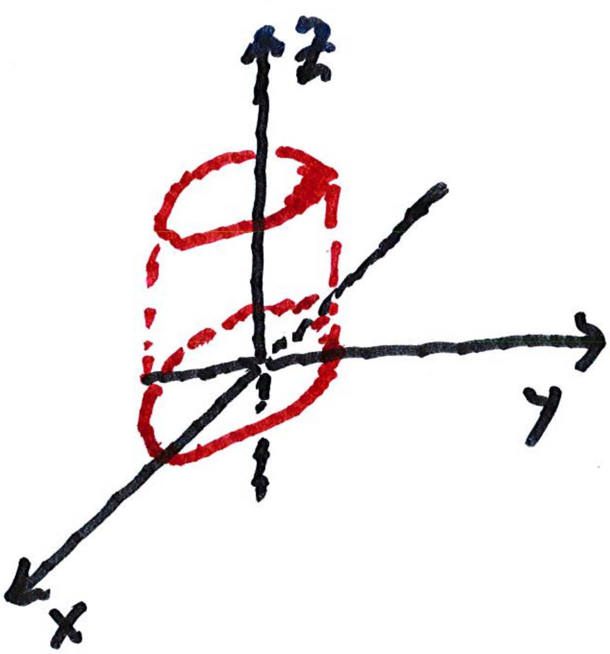
Ex. Sketch $x^2 + y^2 = 4$

Same idea:



$x^2 + y^2 = 4$

(radius = 2)



(ind. of z)

The distance between

$P(x_1, y_1, z_1)$ and $P(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Ex. Find the radius and center of

$$x^2 + y^2 + z^2 + 2x - 4y + 4z = 3$$

$$(x+1)^2 + (y-2)^2 + (z+2)^2$$

$$= 1 + 4 + 4 + 3 = 12$$

$\therefore$ radius $= \sqrt{12}$

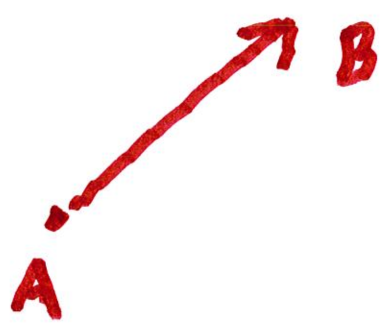
center $= (-1, 2, -2)$

12.2 Vectors

Given points A and B,

we can form a displacement

vector $\overrightarrow{AB}$



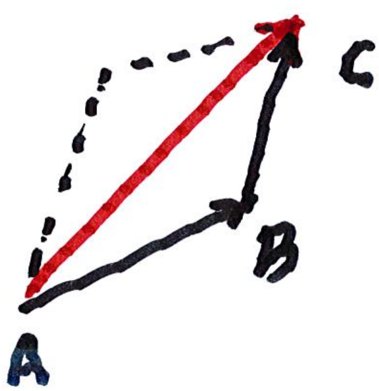
If $\overrightarrow{AB}$ is a translate of $\overrightarrow{CD}$, we consider $\overrightarrow{AB}$ and $\overrightarrow{CD}$ to be the same vector.

A is the initial point and B is the terminal point of $\overrightarrow{AB}$

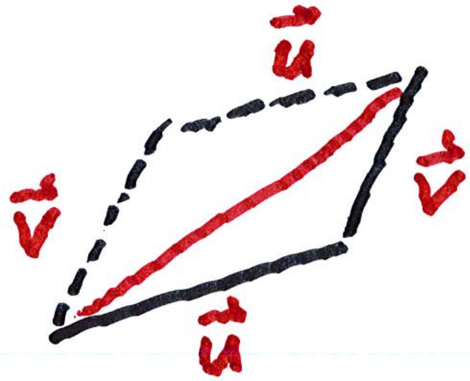
Vector Addition is

defined by the

Parallelogram Law:

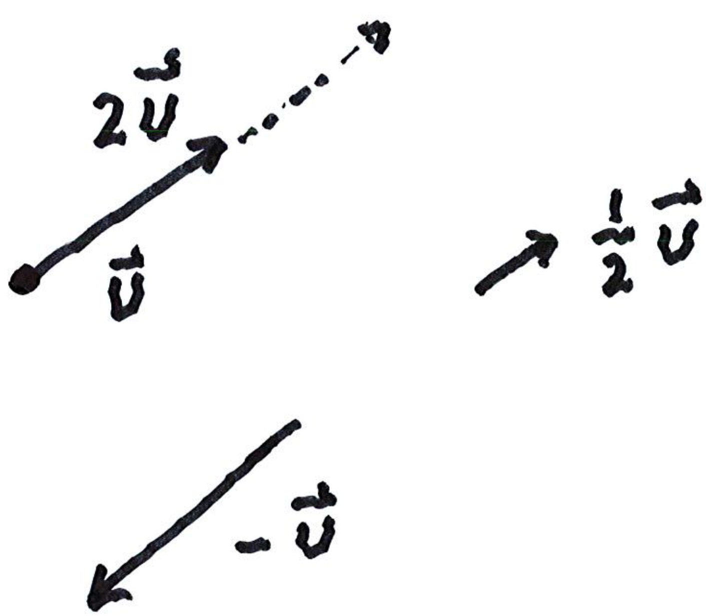


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



$\therefore \vec{u} + \vec{v} = \vec{v} + \vec{u}$

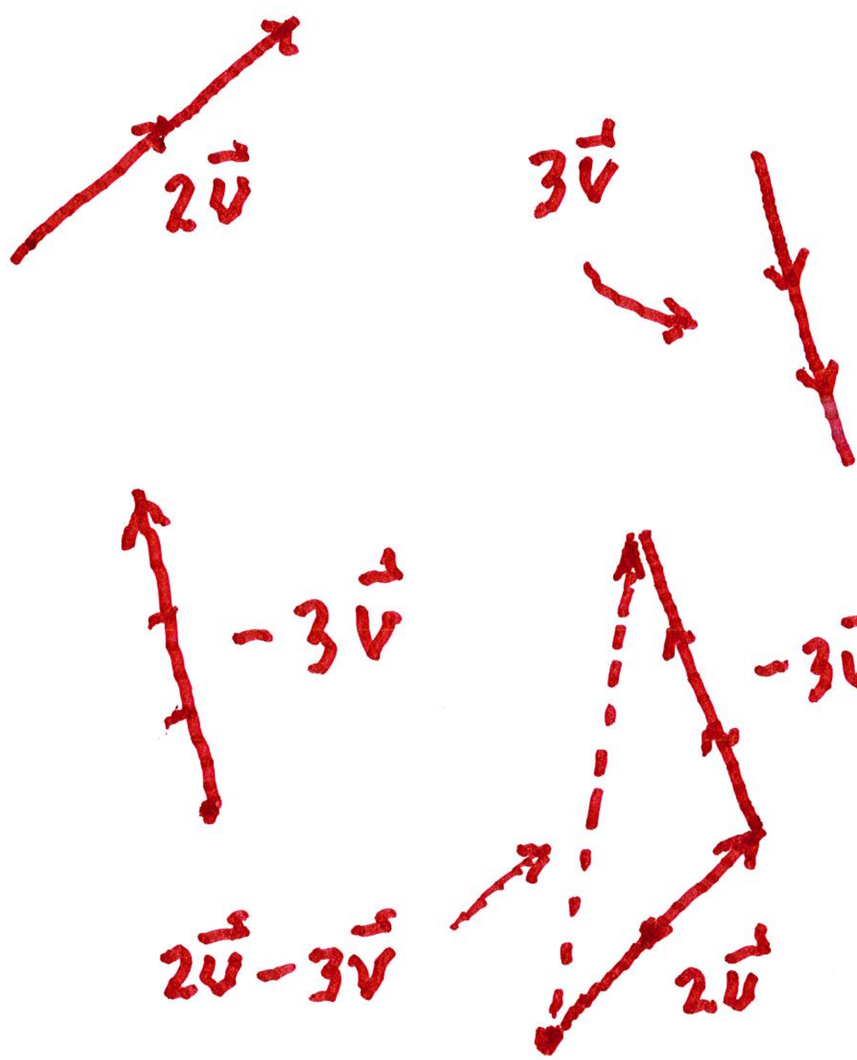
We can multiply $\vec{v}$ by any number c



If $\vec{u}$ is 

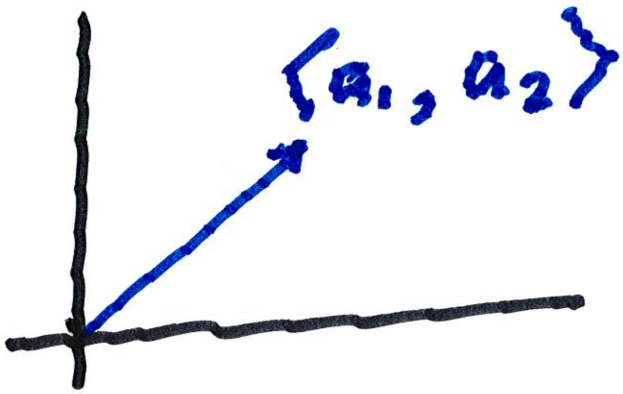
and $\vec{v}$ is  ,

what is $2\vec{u} - 3\vec{v}$



Components

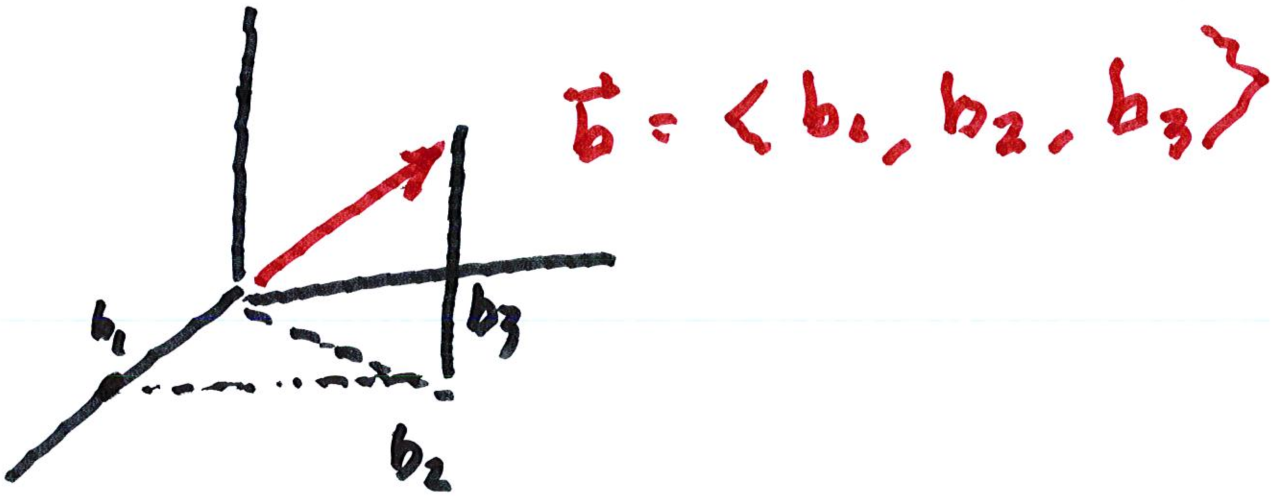
For vectors in $\mathbb{R}^2$:



$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$ absolute value

If $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$



$\vec{a}$ and $\vec{b}$ are called

position vectors.

In $\mathbb{R}^2$, $\vec{a} + \vec{b}$

$= \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$

$= \langle a_1 + b_1, a_2 + b_2 \rangle$

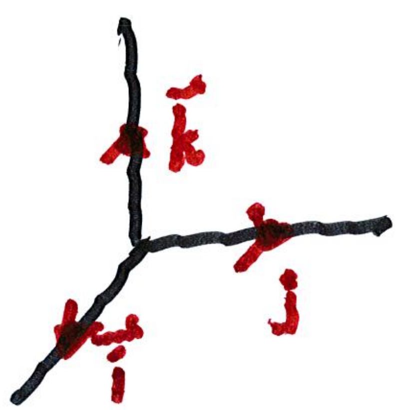
and $c\vec{a} = c\langle a_1, a_2\rangle$

$$= \langle ca_1, ca_2\rangle$$

Standard Basis Vectors

$\vec{i} = \langle 1, 0, 0\rangle \quad \vec{j} = \langle 0, 1, 0\rangle$

and $\vec{k} = \langle 0, 0, 1\rangle$



$\vec{v}$ is a unit vector if

$|\vec{v}| = 1$.

Ex. Find a vector $\vec{w}$ of length 3 that points in the opposite direction from

$\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$$

$$\vec{u} = \frac{1}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

is a unit vector

$$\vec{v} = \frac{3}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

has length 3.

$$\vec{w} = \frac{-3}{\sqrt{14}}\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$$

12.3 Dot Product

Suppose

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ and

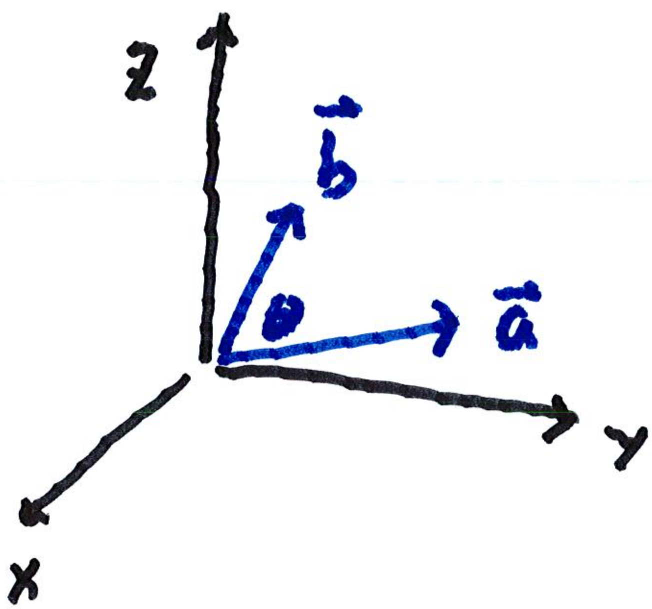
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

The dot product of

$\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Geometric Meaning



Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \quad (1)$$

If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$

then $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

and (1) is still true.

Ex. Find the angle θ between

$\vec{a} = \langle 3, -1 \rangle$ and $\vec{b} = \langle 2, 2 \rangle$

$\vec{a} \cdot \vec{b} = 3 \cdot 2 + (-1) \cdot 2 = 4$

$$|\vec{a}| = \sqrt{9+1} = \sqrt{10}$$

$$|\vec{b}| = \sqrt{4+4} = \sqrt{8}$$

$$\therefore \quad 4 = \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= \sqrt{10}\sqrt{8}\cos\theta$$

$$\rightarrow \quad 4 = \sqrt{80}\cos\theta$$

$$4 = 4\sqrt{5}\cos\theta$$

$$\therefore \cos\theta = \frac{1}{\sqrt{5}} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

Suppose $\vec{a}$ and $\vec{b}$ satisfy

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow |\vec{a}||\vec{b}| \cos\theta = 0$$

$$\therefore \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

$\therefore$ $\vec{a}$ and $\vec{b}$ are perpendicular

Conversely, if $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

Ex. Show $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$

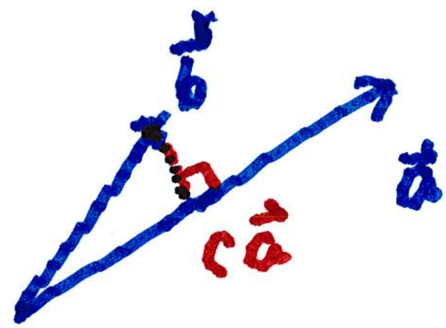
and $\vec{b} = \vec{i} + \vec{j} + \vec{k}$

are perpendicular.

$$\vec{a} \cdot \vec{b} = 2 \cdot 1 - 3 \cdot 1 + 1 \cdot 1 = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Scalar Projections



Find c so that

$\vec{b} - c\vec{a}$ is $\perp$ to $\vec{a}$

$$(\vec{b} - c\vec{a}) \cdot \vec{a} = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} - c\vec{a} \cdot \vec{a} = 0$$

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} = c$$

$$\left(\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2 \right)$$

$$\therefore \ \mathrm{proj}_{\vec{a}} \vec{b} = c\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

This is the vector projection

of $\vec{b}$ onto $\vec{a}$

We can also write

$$c\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\,\frac{\vec{a}}{|\vec{a}|}$$

$\dfrac{\vec{a}}{|\vec{a}|}$ is a unit vector

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ is the scalar projection

It gives the size of $\mathrm{proj}_{\vec{a}}\vec{b}$

in the direction of $\vec{a}$

Ex. Find $\text{comp}_{\vec{a}} \vec{b}$ if

$\vec{a} = 2\vec{i} + \vec{k}$

and $\vec{b} = 3\vec{i} + \vec{j} + 2\vec{k}$

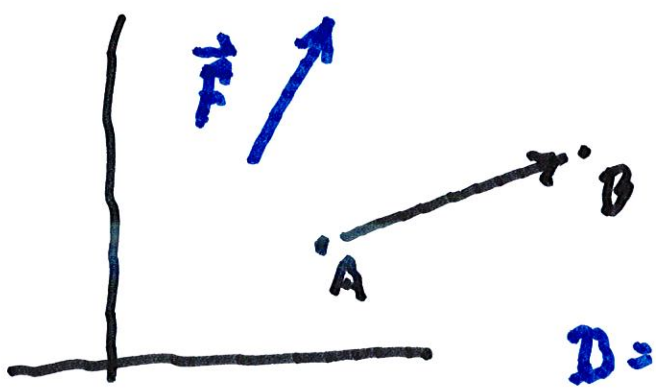
$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{6+2}{\sqrt{4+1}} = \underline{\underline{\frac{8}{\sqrt{5}}}}$$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{8}{5}\left(2\vec{i} + \vec{k}\right)$$

Work

Suppose that there is a constant force $\vec{F}$ and that a particle moves from A to B



$D = \overrightarrow{AB}$ = displacement vector.

The work done by the force is

$$W = \vec{F} \cdot \vec{D}$$

12.4 Cross Product

Suppose $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are

vectors in $\mathbb{R}^3$

We define

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

cross product

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

or $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Ex. Compute $\langle 2, 1, -2 \rangle \times \langle 1, 3 - 1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (-1+6)\vec{i} - (-2+2)\vec{j} + (6-1)\vec{k}$$

$$= 5\vec{i} + 5\vec{k}$$

Fact: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

and $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= (a_2 b_3 - a_3 b_2) a_1 - (a_1 b_3 - a_3 b_1) a_2$$

$$+ (a_1 b_2 - a_2 b_1) a_3 = 0 .$$

$\therefore$ $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

Similarly

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right)$$

Ex. Find a vector $\vec{v}$ that is

$\perp$ to $\langle 1, 2, 3 \rangle$ and $\langle 2, 2, -1 \rangle$

$\vec{a}$ $\qquad\qquad\qquad\qquad$ $\vec{b}$

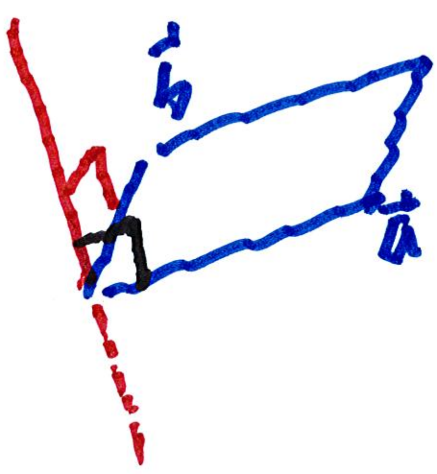
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= -8\vec{i} - (-1-6)\vec{j} + (2-4)\vec{k}$$

$$= \underline{-8\vec{i} + 7\vec{j} - 2\vec{k}} = \vec{v}$$

$\therefore$ $\vec{a} \times \vec{b}$ lies on line that is $\perp$ to $\vec{a}$ and $\vec{b}$



Fact: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

where θ = angle between

$\vec{a}$ and $\vec{b}$.

Also, $|\vec{a} \times \vec{b}|$ = area of

parallelogram spanned

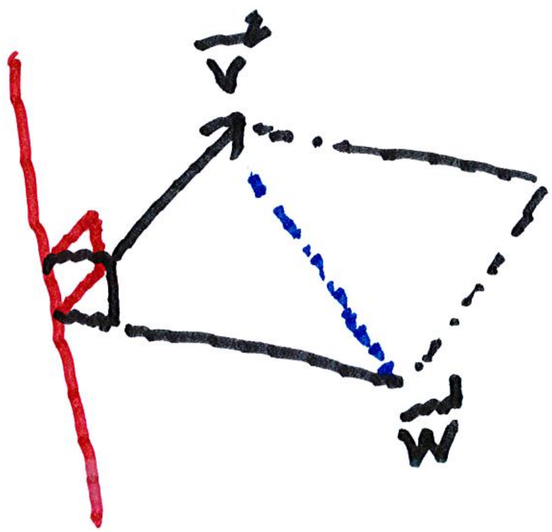
by $\vec{a}$ and $\vec{b}$

Ex. Find the area of the triangle with vertices at $P(2,1,-1)$ $Q(1,1,2)$ and $R(3,2,1)$.

$\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$

$\vec{v} = \langle -1, 0, 3 \rangle$ and

$\vec{w} = \langle 1, 1, 2 \rangle$



$\vec{a} \times \vec{b}$ is $\perp$ to plane spanned by $\vec{v}$ and $\vec{w}$

Area of Δ spanned by $\vec{v}$ and $\vec{w} = \frac{1}{2}\left(\text{Area of Parallelogram}\right)$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -3\vec{i} - (-5)\vec{j} + (-1)\vec{k}$$

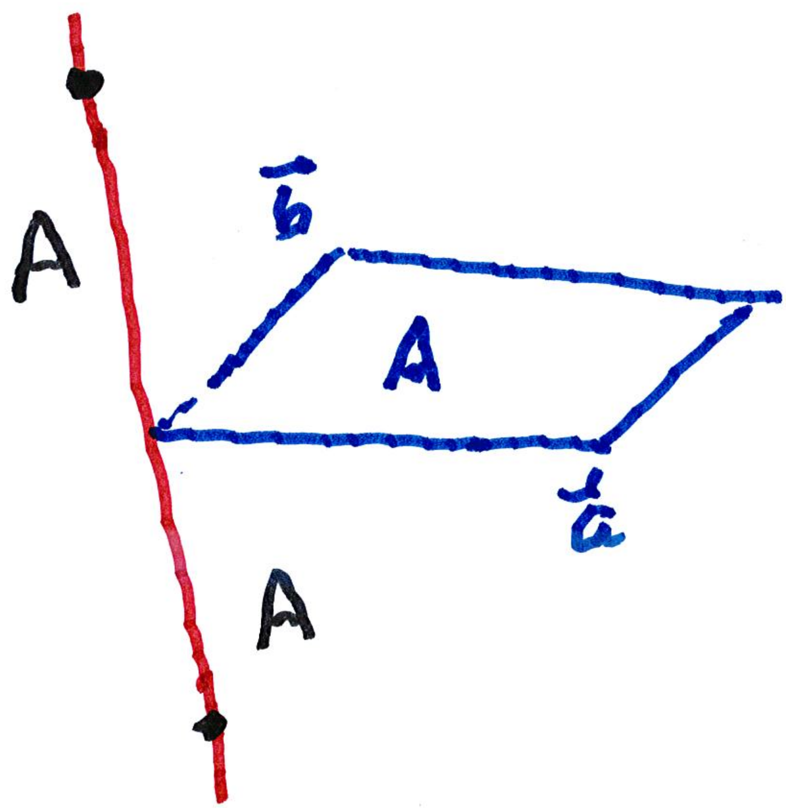
$$= -3\vec{i} + 5\vec{j} - \vec{k}$$

$$|\vec{v} \times \vec{w}| = \sqrt{9 + 25 + 1}$$

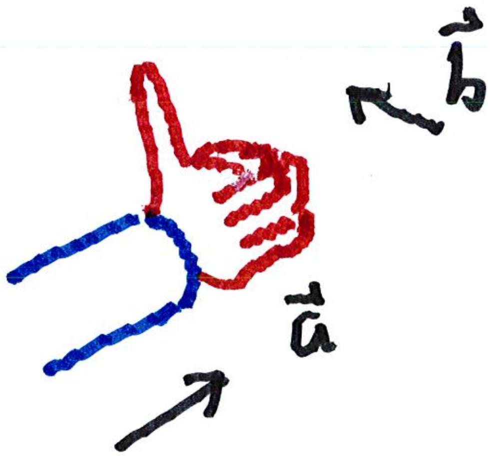
Area of $\Delta = \frac{1}{2}\sqrt{35}$

Now we know $\vec{a} \times \vec{b}$

lies at one of 2 locations



Right Hand Rule



Fingers curl from $\vec{a}$ to $\vec{b}$

$\Rightarrow$ $\vec{a} \times \vec{b}$ points in

direction of thumb

Triple Product

Given vectors

$\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} \cdot (\vec{b} \times \vec{c})$ (the triple product)

equals volume of parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$