12.4 Cross Product

Suppose $\vec{a} = \langle a_1, a_2, a_3 \rangle$

and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are

vectors in $\mathbb{R}^3$

We define

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2 ,\ a_3 b_1 - a_1 b_3 ,\ a_1 b_2 - a_2 b_1 \rangle$$

cross product

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

or $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Ex. Compute $\langle 2, 1, -2 \rangle \times \langle 1, 3 -1 \rangle$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (-1+6)\vec{i} - (-2+2)\vec{j} + (6-1)\vec{k}$$

$$= 5\vec{i} + 5\vec{k}$$

Fact: $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

and $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$\{\vec{a} \times b\} \cdot \vec{a}$

$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$

$= \{a_2 b_3 - a_3 b_2\} a_1 - \{a_1 b_3 - a_3 b_1\} a_2$

$+ \{a_1 b_2 - a_2 b_1\} a_3 = 0 .$

$\therefore \vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$

Similarly

$\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$= \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \right)$$

Ex. Find a vector $\vec{v}$ that is $\perp$ to $\underbrace{\langle 1, 2, 3 \rangle}_{\vec{a}}$ and $\underbrace{\langle 2, 2, -1 \rangle}_{\vec{b}}$

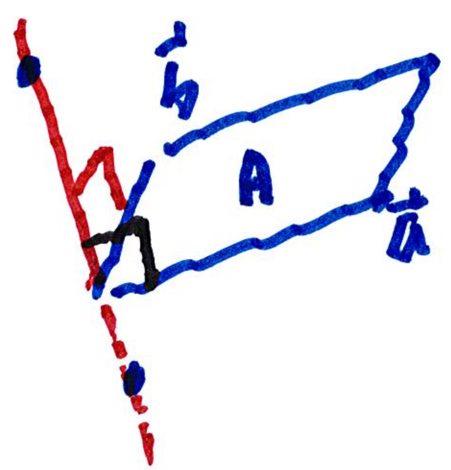
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= -8\vec{i} - (-1-6)\vec{j} + (2-4)\vec{k}$$

$$= \underline{-8\vec{i} + 7\vec{j} - 2\vec{k}} = \vec{v}$$

$\therefore$ $\vec{a} \times \vec{b}$ lies on line that is $\perp$ to $\vec{a}$ and $\vec{b}$



Fact: $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

where θ = angle between

$\vec{a}$ and $\vec{b}$.

Also, $|\vec{a} \times \vec{b}|$ = area of

parallelogram spanned

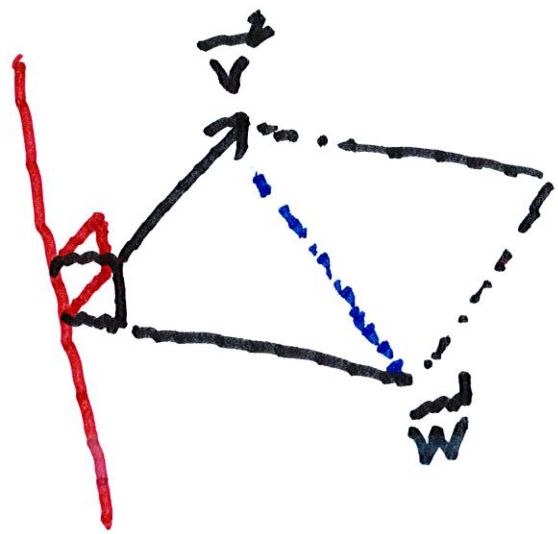
by $\vec{a}$ and $\vec{b}$

Ex. Find the area of the triangle with vertices at $P(2,1,-1)$ $Q(1,1,2)$ and $R(3,2,1)$.

$\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{PR}$

$\vec{v} = \langle -1, 0, 3 \rangle$ and

$\vec{w} = \langle 1, 1, 2 \rangle$



$\vec{a} \times \vec{b}$ is $\perp$ to plane spanned by $\vec{v}$ and $\vec{w}$

Area of Δ spanned by $\vec{v}$ and $\vec{w}$ $= \frac{1}{2}$ (Area of Parallelogram)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= -3\vec{i} - (-5)\vec{j} + (-1)\vec{k}$$

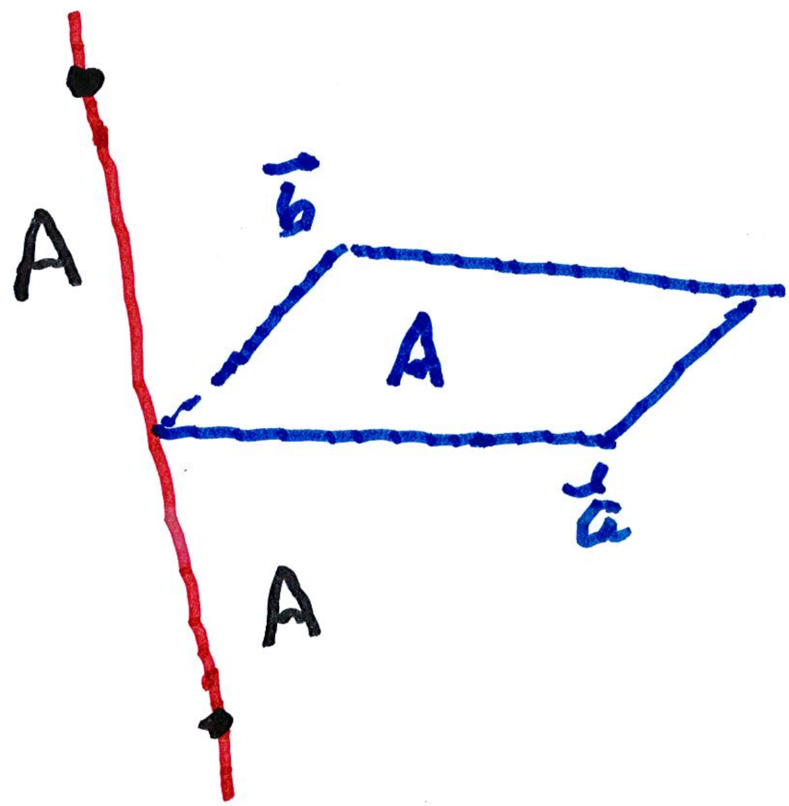
$$= -3\vec{i} + 5\vec{j} - \vec{k}$$

$$|\vec{v} \times \vec{w}| = \sqrt{9 + 25 + 1}$$

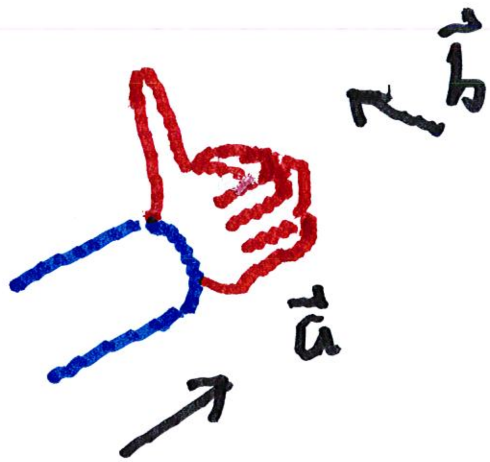
Area of $\Delta = \frac{1}{2}\sqrt{35}$

Now we know $\vec{a} \times \vec{b}$

lies at one of 2 locations



Right Hand Rule



Fingers curl from $\vec{a}$ to $\vec{b}$

$\Rightarrow$ $\vec{a} \times \vec{b}$ points in

direction of thumb

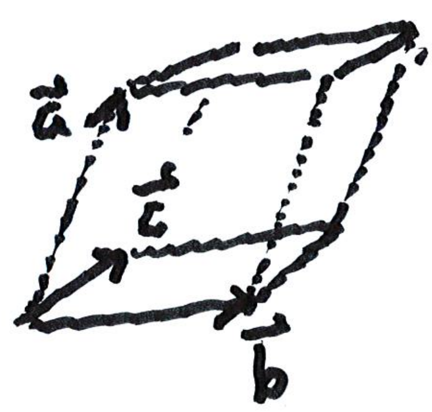
Triple Product

Given vectors

$\vec{a}$, $\vec{b}$ and $\vec{c}$,

$\vec{a} \cdot (\vec{b} \times \vec{c})$ (the triple product)

equals volume of parallelepiped spanned by $\vec{a}$, $\vec{b}$ and $\vec{c}$

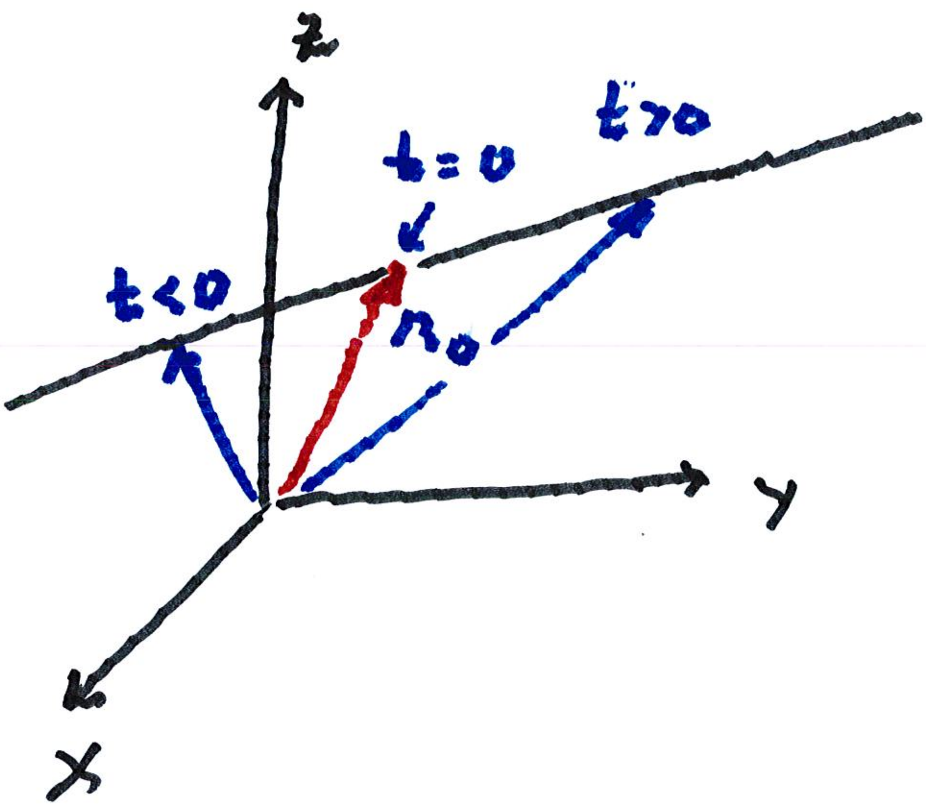


12.5 Lines and Planes in Space

Any line L is determined by a point $\vec{r}_0$ in L and a vector $\vec{v}$ that is parallel to L. Then L

is $\quad \vec{r}(t) = \vec{r}_0 + t\vec{v}$

These are the vector equations of L.



If $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

and $r(t) = \langle x(t), y(t), z(t) \rangle$

and $\vec{v} = \langle a, b, c \rangle$,

then

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

or

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

These are called the

parametric equations of L.

Find the vector equations

and parametric equations

of the line L through (2, 1, −1)

that is parallel to ⟨1, 2, 3⟩

Set $\vec{r}_0 = \langle 2, 1, -1 \rangle$ and

$\vec{v} = \langle 1, 2, 3 \rangle$

$\vec{r} = \langle 2, 1, -1 \rangle + t\langle 1, 2, 3 \rangle$

(vector equations)

and

$$x = 2 + t, \quad y = 1 + 2t, \quad z = -1 + 3t$$

(parametric equations)

Another way of describing

L is to eliminate the parameter

t:

Solving for t in all 3

parametric equations,

$$\frac{x-x_0}{a} = t \qquad \frac{y-y_0}{b} = t \qquad \frac{z-z_0}{c} = t$$

Since all equations are equal,

we get

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

These are called the

symmetric equations of L

Ex. Find the vector equations, the parametric equations and the symmetric equations of the line L that passes through $\langle 2, 3, -1 \rangle$ and $\langle 1, 1, 2 \rangle$.

$$\vec{v} = \langle 1, 1, 2 \rangle - \langle 2, 3, -1 \rangle$$

$$\vec{v} = \langle -1, -2, 3 \rangle$$

Also, set $\vec{r}_0 = \langle 2, 3, -1 \rangle$

$$\vec{r} = \langle 2, 3, -1 \rangle + t \langle -1, -2, 3 \rangle$$

vec. eqns

$$x = 2 - t, \quad y = 3 - 2t, \quad z = -1 + 3t$$

par. eqns

$$\frac{x-2}{-1} = \frac{y-3}{-2} = \frac{z+1}{3}$$

sym. eq'ns:

Ex. At what point does the above line L pass through the xz-plane?

xz-plane is $\underline{y = 0.}$

$\therefore \quad 0 = 3 - 2t \rightarrow t = \frac{3}{2}$

In parametric equations

$x = 2 - \frac{3}{2} = \frac{1}{2}$ $\qquad z = -1 + \frac{9}{2}$

$= \frac{7}{2}$

$\therefore$ point is $\left\langle \frac{1}{2}, 0, \frac{7}{2} \right\rangle$

Two lines L_1 and L_2

are skew if they are

not parallel and don't

intersect. Show that

L_1 : $x = 2+t$, $y = 1-2t$, $z = 3+3t$

L_2 : $x = 3-t$, $4-4t$, $z = 1+2t$

are skew:

First, the direction vectors

$\langle 1, -2, 3 \rangle$ and $\langle -1, -4, 2 \rangle$

are not parallel.

We write L_2 as

$x = 3 - s, \ y = 4 - 4s, \ z = 1 + 2s$

A point of intersection

would satisy

$$3 - s = 2 + t \qquad E_1$$

$$4 - 4s = 1 - 2t \qquad E_2$$

$$1 + 2s = 3 + 3t \qquad E_3$$

Now eliminate t in E_1, E_2

$2E_1 + E_2$: $2(3-s) + (4-4s) = 5$

or $10 - 6s = 5$

or $s = \dfrac{5}{6}$

Now eliminate t in E_2, E_3

$3E_2 + 2E_3$: $3(4-4s) + 2(1+2s) = 9$

or $14 - 8s = 9$

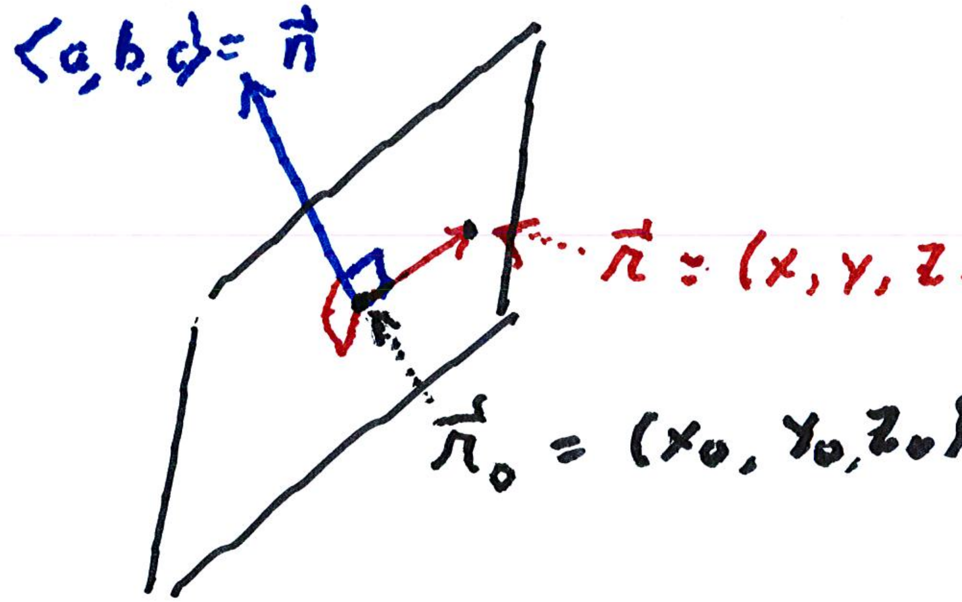
or $s = \frac{5}{8}$

$\therefore$ Lines don't intersect.

$\therefore$ Lines are skew

Planes

A plane $\mathcal{P}$ is determined by a point $\vec{r}_0$ in the plane and a normal vector that is orthogonal (perpendicular) to $\mathcal{P}$. If $\vec{r}$ is any point in $\mathcal{P}$, then $\vec{r} - \vec{r}_0$ lies in $\mathcal{P}$.



$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$0 = \langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle$$

and so, $\vec{n}$ is $\perp$ to $\vec{r} - \vec{r}_0$

$$\therefore \quad \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad (*)$$

$$\text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

holds for every r in P.

This is called the

vector equation of the plane

If $\vec{n} = \langle a, b, c \rangle$

and $r = \langle x, y, z \rangle$ and

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$,

then (*) becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

This is called the scalar equation of P

Ex. Find an equation of the plane P through the point $(1, 4, 2)$ with normal vector $\vec{n} = \langle 3, 2, 5 \rangle$.

$\vec{r} = \langle x, y, z \rangle$

The scalar eq'n is

$$3(x-1) + 2(y-4) + 5(z-2) = 0$$

or $3x + 2y + 5z = 3 + 8 + 10$

or $3x + 2y + 5z = 21$

This last equation is

called ~~the~~ a linear equation of θ.

Ex. Find the linear equation of the plane P that passes through $P(1, 2, -2)$, $Q(2, 1, 1)$ and $R(1, 3, 4)$.

Note that $\overrightarrow{PQ} = \langle 2-1, 1-2, 1-(-2) \rangle$

$= \langle 1, -1, 3 \rangle$ is in P.

$\vec{n}$

So does $\overrightarrow{PR} = \langle 1-1,\ 3-2,\ 4-(-2) \rangle$

$= \langle 0, 1, 6 \rangle = \vec{b}$

Hence $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 0 & 1 & 6 \end{vmatrix}$

$= (-6-3)\vec{i} - (6-0)\vec{j} + (1-0)\vec{k}$

$= -9\vec{i} - 6\vec{j} + \vec{k}$ is $\perp$ to P.

Set $\vec{n} = \langle -9, -6, 1 \rangle$

and $\vec{r}_0 = \langle 1, 2, -2 \rangle$

The vector equation is

$\vec{n} \cdot (\vec{r} - \vec{r}_0)$ or

$-9(x-1) - 6(y-2) + (z+2) = 0$

or $\underline{-9x - 6y + z = -9 - 12 - 2 = -23}$