14.2 Limits and Continuity

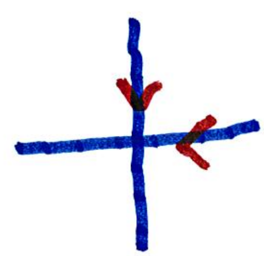
Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

What is $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

Does the limit exist ?

Along the x-axis

$f(x,0) = 0$

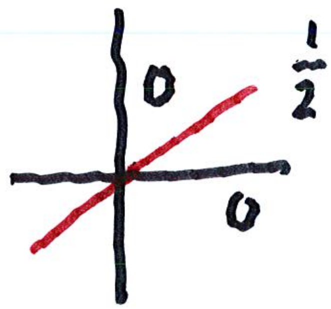


Along the y-axis ,

$$f(0,y) = \frac{0 \cdot y}{y^2} = 0$$

So, a good guess would be

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$$



Let's try

$y = x$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \text{ does not exist}$$

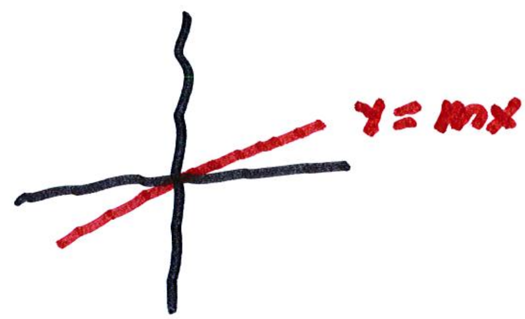
D. N. E.

The value of

~~What about~~ $\lim_{(x,y)\to(0,0)} f(x,y)$

should be the same for

<u>all</u> lines $y = mx$

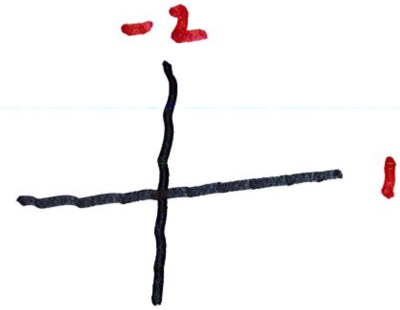


Ex. Find $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - 2y^2}{x^2 + y^2}$

On x-axis $\dfrac{x^2 - 0}{x^2} = 1$

On y-axis $\dfrac{0 - 2y^2}{0 + y^2} = -2$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2} \quad \text{D.N.E.}$$



What is the correct definition?

Def'n. Suppose $f(x,y)$ is defined for all (x,y) near, (but not for (a,b)) We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

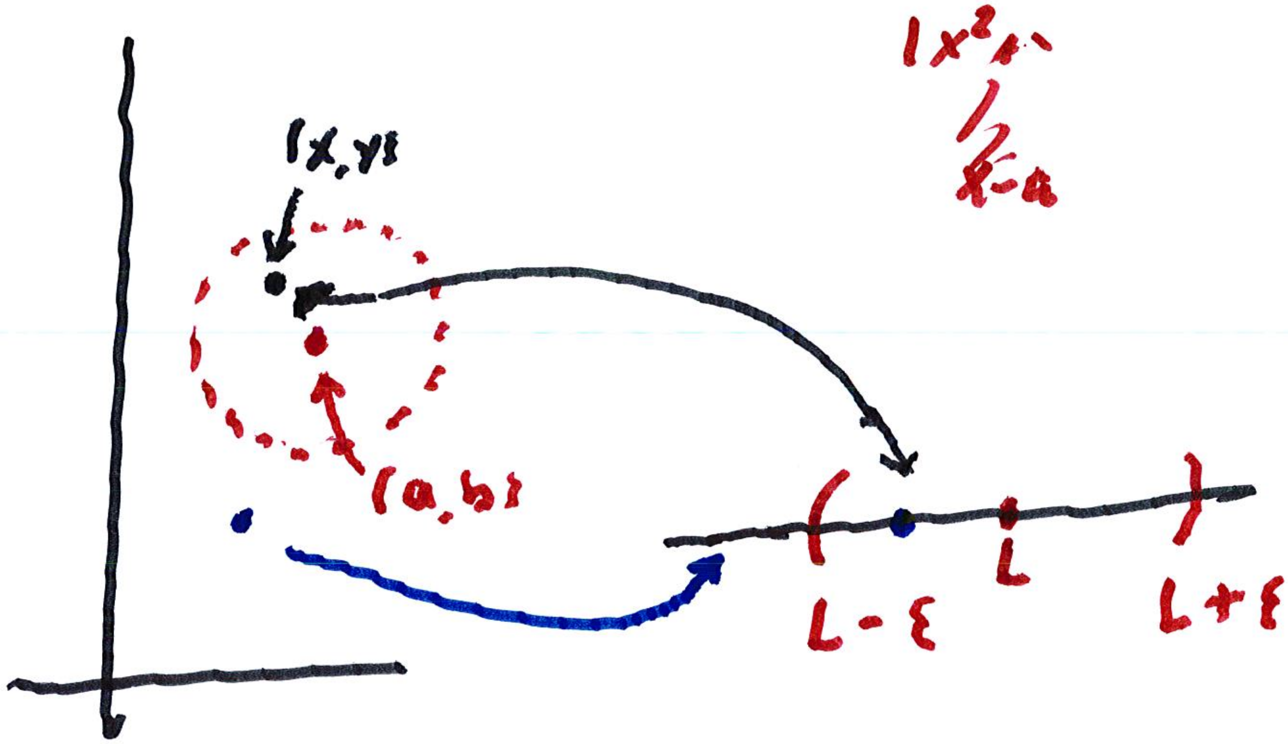
if for every number $\varepsilon > 0$,

there is a corresponding

number $\delta > 0$, so that <u>if</u>

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

then $|f(x,y) - L| < \varepsilon$



You have to show that if

(x, y) is within δ of (a, b),

then $f(x, y)$ is within ε of L.

Ex. $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$ $\quad (x,y) \neq (0,0)$

Show $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

$$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}$$

$$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$$

$$\therefore \quad |x^2y| \leq |x|^2|y|$$

$$\leq \left(\sqrt{x^2+y^2}\right)^3$$

$$\left| \frac{3x^2y}{x^2+y^2} \right| = \frac{|3x^2y|}{x^2+y^2}$$

$$\leq \frac{3\left(\sqrt{x^2+y^2}\right)^3}{x^2+y^2}$$

$$= 3\sqrt{x^2+y^2}$$

or $$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| \leq 3\sqrt{x^2+y^2}.$$

if $(x,y) \neq (0,0)$

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$.

If $0 < \sqrt{x^2+y^2} < \delta$

then $\left| \frac{3x^2y}{x^2+y^2} - 0 \right|$

$\leq 3\sqrt{x^2+y^2} < 3\delta = \varepsilon.$

$\therefore$ Using $\delta = \frac{\varepsilon}{3}$, we've shown

that if $0 < \sqrt{x^2+y^2} < \delta$, then

$\left| \frac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$

How can we show that some limits <u>do</u> exist?

One can show $\lim_{(x,y)\to(a,b)} x = a$

and $\lim_{(x,y)\to(a,b)} y = b$.

Moreover, the same limit laws still apply:

Does $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$ exist ?

Try x-axis $\quad \dfrac{x^2 \cdot 0}{x^4} = 0$

Try y-axis $\quad \dfrac{0 \cdot y}{y^2} = 0$

Try $y = mx$ (any line)

$$\frac{x^2 (mx)}{x^4 + m^2 x^2}$$

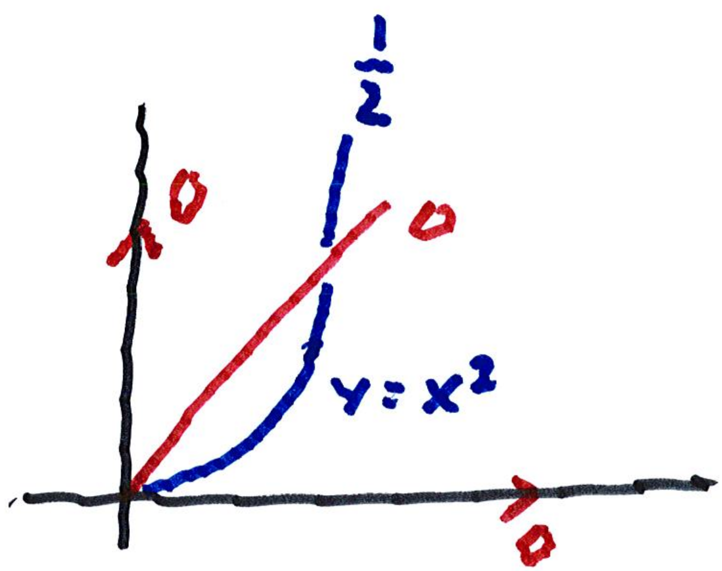
$$= \frac{mx}{x^2 + m^2} \to \frac{m \cdot 0}{0 + m^2} = 0$$

Thus it would appear that

$$\lim \frac{x^2 y}{x^4 + y^2} = 0.$$

But if we try $y = x^2$,

$$\frac{x^2 \cdot x^2}{x^4 + x^4} = \frac{1}{2}$$



So, to show $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

we must show $f(x,y) \to L$

as $(x,y) \to (a,b)$

on any curve or line.

Useful Idea: If $f(x,y) \to L_1$

as $(x,y) \to (a,b)$

along C_1 (a curve)

and if $f(x,y) \to L_2$

as $(x,y) \to (a,b)$ along C_2, and

if $L_1 \neq L_2$, then $\lim f(x,y)$ D.N.E.

Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$

and $\lim_{(x,y)\to(a,b)} g(x,y) = L_2$.

then (1) $\lim \{f+g\} = L_1 + L_2$

(2) $\lim cf = c L_1$

(3) $\lim fg = L_1 L_2$

(4) $\lim f/g \to L_1/L_2$

provided $L_2 \neq 0$.

Continuity.

Def'n. A function f of 2 variables x and y is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is continuous on D if f is continuous at every point (a,b) in D.

It follows that $f(x,y) = x$

and $g(x,y) = y$ are continuous

on any D. Also

<u>any</u> polynomial $P(x,y)$

$$= \sum_{j,k=1}^{N} C_{jk}\, x^j y^k$$ is continous

Also, any rational fcn.

$$R(x,y) = \frac{P(x,y)}{Q(x,y)}$$ is

continuous at any point

(x,y) as long as $Q(x,y) \neq 0$.

In addition, if

$\lim f_1(x,y) = L$, and

$\lim f_3(x,y) = L$, and

$f_1(x,y) \leq f_2(x,y) \leq f_3(x,y)$

then $\lim\limits_{(x,y)\to(a,b)} f_2(x,y) = L$

Squeeze Thm.

Functions of 3 variables

Def'n of limit.

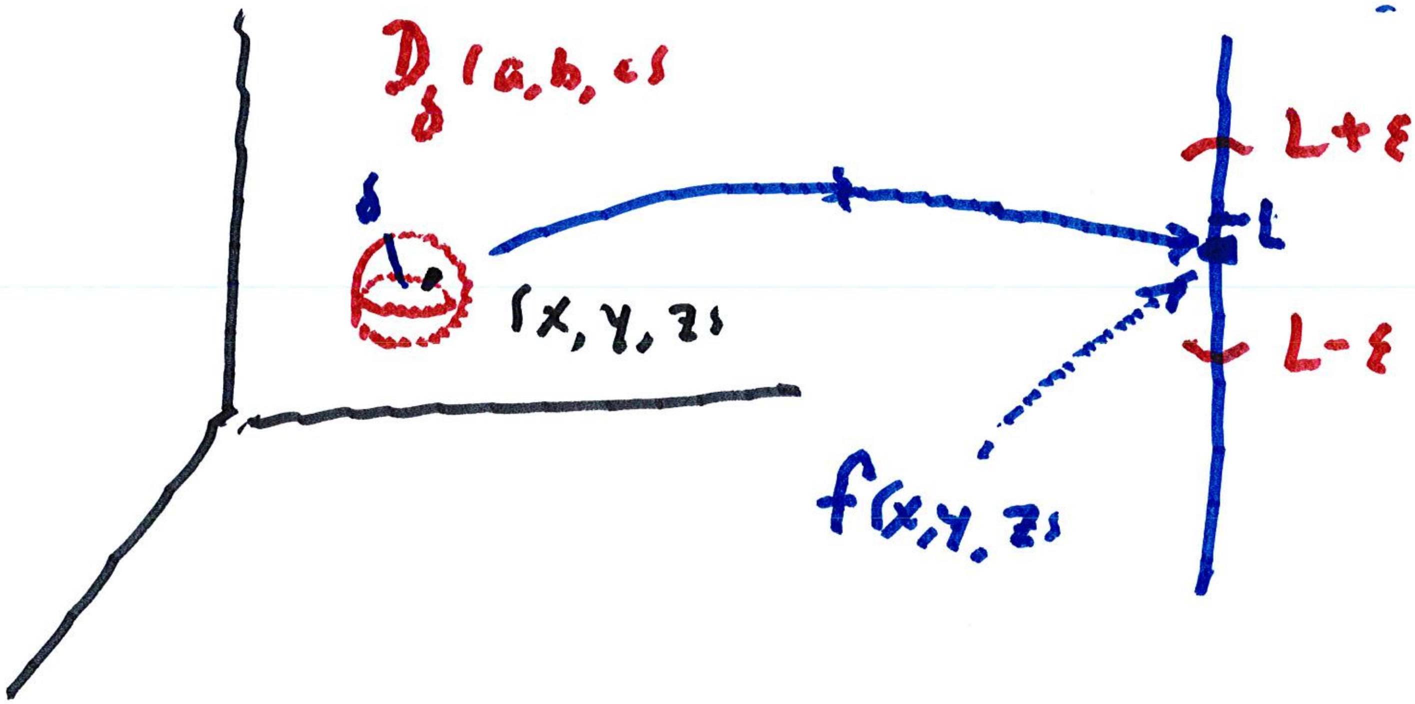
We say $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$

if for ~~all~~ every $\varepsilon > 0$, there is

a $\delta > 0$, so if

$$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta,$$

then $|f(x,y,z) - L| < \varepsilon.$



Ex. $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + 2xz}{x^2+y^2+z^2}$ DNE

but $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{3xy^2 + z^3}{x^2+y^2+z^2}$ Does Exist.

$= 0$