14.3 Partial Derivatives

In calculus of 1 variable,

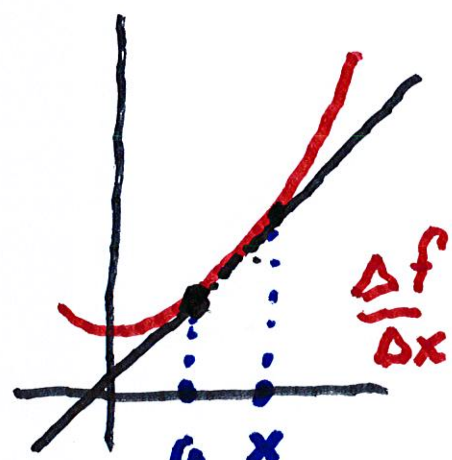
the derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

tells how a function $f(x)$

is changing

when x is near a



How does $f(x,y)$ change when

we only allow x to change?

Thus we define

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

And if we only allow y to change?

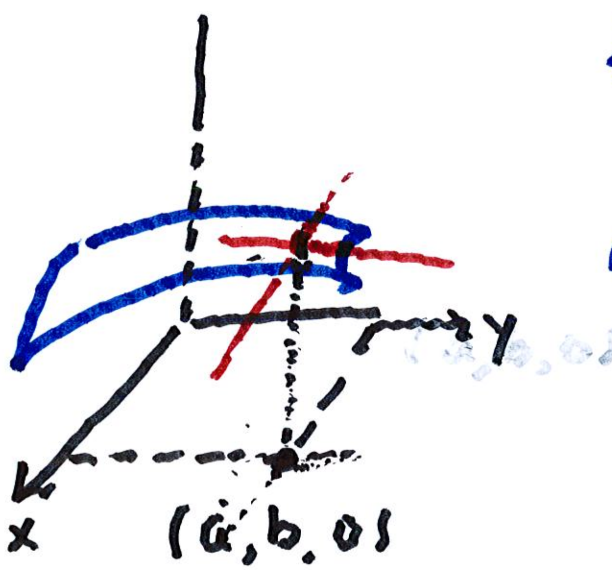
$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

If we now let the point (a, b)

to vary:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(a, b) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$f_x(x, y)$ and $f_y(x, y)$

are called

partial derivatives

of f at (x, y)

There are many alternative

notations for

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y)$$

$$= \frac{\partial z}{\partial x} = D_1 f = D_x f$$

and

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y)$$

$$= \frac{\partial z}{\partial y} = D_1 f = D_y f$$

Rule for finding Partial Derivatives

of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Ex. 1 If $f(x,y) = x^4 + x^2y^3 - 2y^3$

For f_x, keep y constant.

Find $f_x(1,2)$

$$f_x(x,y) = 4x^3 + 2xy^3 = 4\cdot 1 + 2\cdot 1\cdot 8$$

$$= 20$$

For f_y, keep x constant.

Find $f_y(1,2)$

$$f_y(1,2) = 3x^2y^2 - 6y^2 = 3\cdot 1\cdot 4 - 6\cdot 4$$

$$= -12$$

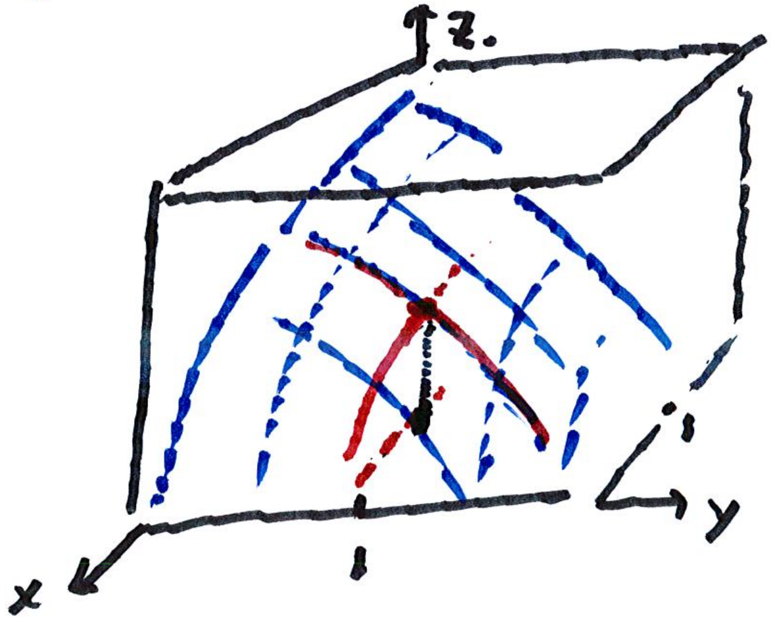
Ex. If $f(x,y) = 4 - x^2 - 2y^2$,

find $f_x(1,1)$ and $f_y(1,1)$.

$f_x = -2x = -2$

$f_y = -4y = -4$

f_x



$f_x(1,1) = -2$ = rate of changing

as x changes at (1,1)

Similarly,

$f_y(1,1) = -4$ = rate of changing

at $(1,1)$

we can use all of the rules

for differentiation to

compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

if $f(x, y) = \cos\left(\frac{x^2}{1+y^3}\right)$

$$\frac{\partial f}{\partial x} = -\sin\left(\frac{x^2}{1+y^3}\right) \cdot \frac{2x}{1+y^3}$$

$$\frac{\partial f}{\partial y} = -\sin\left(\frac{x^2}{1+y^3}\right)\left(\frac{-x^2}{(1+y^3)^2}\right) 3y^2$$

Use implicit differentiation

to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

if z is defined implicitly by

$$x^2 + 2y^3 + z^4 - x^2y + z + 1 = 0$$

First with respect to x:

$$2x + \quad + 4z^3 \frac{\partial z}{\partial x} - 2xy + \frac{\partial z}{\partial x} = 0$$

Solve for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x}\left(1+4z^3\right) = 2xy - 2x$$

$$\therefore \frac{\partial z}{\partial x} = \frac{2xy-2x}{1+4z^3}$$

Now with respect to y :

$$6y^2 + 4z^3\frac{\partial z}{\partial y} - x^2 + \frac{\partial z}{\partial y} = 0$$

Solve for $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial y}\left(1 + 4z^3\right) = x^2 - 6y^2$$

$$\rightarrow \quad \frac{\partial z}{\partial y} = \frac{x^2 - 6y^2}{1+4z^3}$$

The problem is that we

don't know what z is

For functions of 3 variables:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

To differentiate, keep y and z fixed.

Ex. If $f(x, y, z) = \dfrac{xy^2}{x+y+z}$,

find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \frac{(x+y+z)\,y^2 - xy^2 \cdot 1}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y+z)\,2xy - xy^2 \cdot 1}{(x+y+z)^2}$$

$$\frac{\partial f}{\partial z} = \frac{-xy^2}{(x+y+z)^2}$$

Ex. Find first derivatives

if $w = x^{(y/z)}$

$x = e^{\ln x}$

$\therefore \quad x^{(y/z)} = \left(e^{\ln x}\right)^{y/z}$

$$= e^{\frac{y \ln x}{z}}$$

$$\therefore \quad \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left\{ e^{\frac{y \ln x}{z}} \right\}$$

$$= e^{\frac{y \ln x}{z}} \cdot y/xz$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(e^{\frac{y \ln x}{z}} \right)$$

$$= e^{\frac{y \ln x}{z}} \cdot \frac{\ln x}{z}$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(e^{\frac{y \ln x}{z}} \right)$$

$$= e^{\frac{y \ln x}{z}} \cdot \left(\frac{-y \ln x}{z^2} \right)$$

Higher Derivatives

If $z = f(x,y)$, then

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

Similar formulas for $(f_x)_y$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Ex. If $f(x,y) = x^3 + x^2y^4 + 3y^2$

$$f_x = 3x^2 + 2xy^4$$

$$(f_x)_y = 8xy^3$$

$$(f_x)_x = 6x + 2y^4$$

$$f_y = 4x^2y^3 + 6y$$

$$(f_y)_x = 8xy^3$$

$$(f_y)_y = 12x^2y^2 + 6$$

Note that

$$(f_x)_y = 8xy^3 = (f_y)_x$$

Thm. If f_{xy} and f_{yx} are continuous in a disk D, then

$$f_{xy} = f_{yx}$$

Ex. Compute f_{xyz} if

$$f(x,y,z) = x^2y - y^2z^3x^2$$

$$f_x = 2xy - 2xy^2z^3$$

$$f_{xy} = 2x - 4xyz^3$$

$$f_{xyz} = -12xyz^2$$

Composition

Suppose $g(t)$ is continuous

at T and

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = T.$$

Then

$$\lim_{(x,y,z)=(a,b,c)} \left(g(f(x,y,z))\right) = g(T).$$

Ex. Compute

$$\lim_{(x,y)\to(2,3)} e^{\left(\frac{xy}{x+y}\right)}$$

e^t is cont. at any t

$$\lim = e^{\frac{2\cdot 3}{2+3}} = e^{6/5}.$$

$z^5 + 2z - 5 = 0$

$z^5 + 2xz + 5 = 0$ $z(x)$ is defined implicitly

$z(x, y)$

$-2x^2 z$ $-4x \cdot z - 2x^2 \frac{\partial z}{\partial x}$