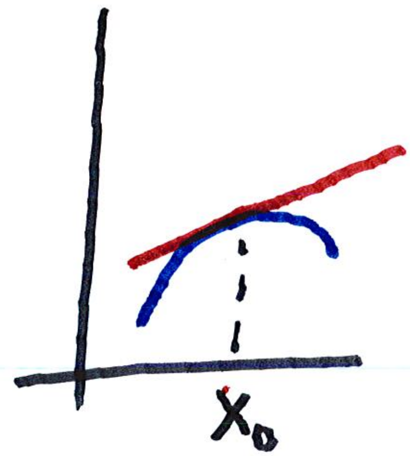
14.4 Tangent Planes and Approximations

Suppose we are given a curve $y = f(x)$. From 1-variable calculus that if a curve $y = f(x)$ passes through (x_0, y_0), then the line that best approximates the curve is

$$y - y_0 = f'(x_0)(x - x_0)$$



Now suppose are given a surface

$z = f(x, y)$ that passes through (x_0, y_0, z_0). We want to know which plane best approximates the surface.

A plane can be expressed as

(1) $$z - z_0 = a(x - x_0) + b(y - y_0)$$

If we fix $y = y_0$ and allow

x to vary, then the curve

becomes $z - z_0 = f(x, y_0)$

and the slope of the

is $\frac{\partial f}{\partial x}(x_0, y_0)$

If we fix $x = x_0$, and allow y to vary, then the curve becomes $z - z_0 = f(x_0, y)$,

and the slope of the line is $\frac{\partial f}{\partial y}(x_0, y_0)$. It makes sense that the coefficients $\underline{a}$ and $\underline{b}$ in (1) are $a = \frac{\partial f}{\partial x}(x_0, y_0)$

and $b = \frac{\partial f}{\partial y}(x_0, y_0)$

Hence the plane that

best fits the surface $z = f(x, y)$

is

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Since $z_0 = f(x_0, y_0)$ we get

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Ex. Find the plane that approximates $z = \underbrace{x^3 - xy^2 + y^3}_{f(x,y)}$ at $(1, 2, 5)$.

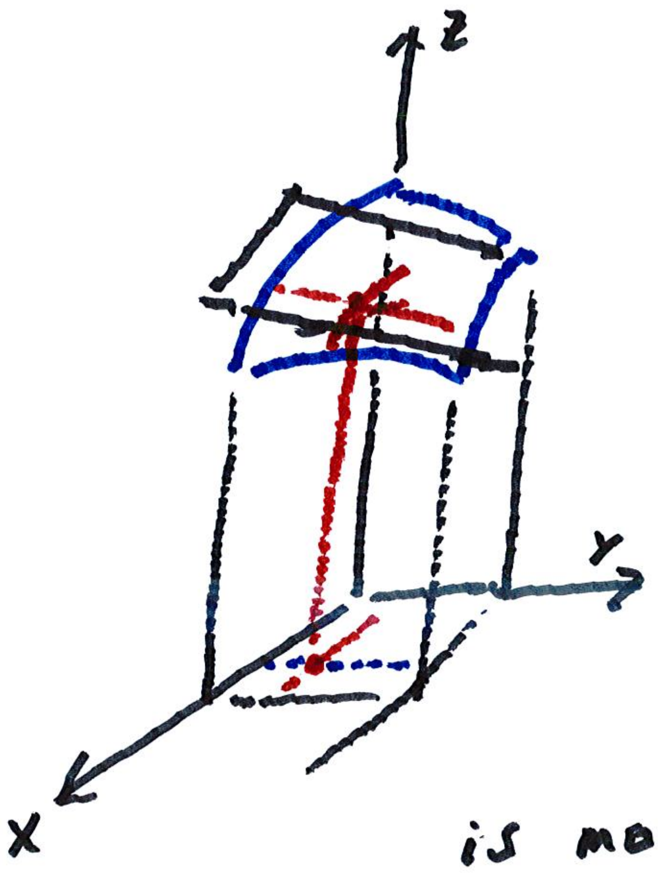
$$\frac{\partial f}{\partial x}(1,2) = 3x^2 - y^2 = 3 - 4 = -1$$

$$\frac{\partial f}{\partial y}(1,2) = -2xy + 3y^2 = -4 + 12 = 8$$

$\rightarrow$ $z - 5 = -1(x-1) + 8(y-2)$

One can also rewrite this as

$$z = -x + 8y - 10$$



As (x, y) gets closer to (x_0, y_0) and we zoom in the surface is more like the plane.

$\frac{\partial f}{\partial x}(x_0, y_0)$ is the rate of

change in the x-direction

and

$\frac{\partial f}{\partial y}(x_0, y_0)$ is the rate of

change in the y-direction.

Find the equation of the plane

that approximates

$z = 3x^2 - 2xy^3 + y^3$ at $(2,1)$

$$\frac{\partial f}{\partial x} = 6x - 2y^3 = 12 - 2 = \underline{10}$$

$$\frac{\partial f}{\partial y} = -6xy^2 + 3y^2 = -12 + 3 = \underline{-9}$$

Also. $f(2,1) = 12 - 4 + 1 = 9$

$$z - 9 = 10(x-2) - 9(y-1)$$

This plane is called the

tangent plane at

(2, 1, 9)

In general the tangent plane

is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

This is called the linear

approximation or the

tangent approximation of

f at (x_0, y_0). This is a

good approximation if

$\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$

are continuous near (x_0, y_0).

If those functions are <u>not</u>

continuous, it may be a poor

approximation.

Ex. $f(x,y) = \dfrac{xy}{x^2+y^2}$, $f(0,0) = 0$

We can use the approximation

to measure changes in Δf

$$f(x,y) - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a)$$

$$+ \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$f_x = \frac{(x^2+1)(3x^2-y^2) - 2x(x^3 - xy^2 + y^2)}{(x^2+1)^2}$$

$$= \frac{2(-1) - 2(1 - 4 + 4)}{2^2}$$

$$=$$

$$= \frac{-2 - 2 + 8 - 8}{4} = -1$$

$$f_y = \frac{(x^2+1)(-2xy+2y) - 2x(x^3-xy^2+y^2)}{(x^2+1)^2}$$

$$= \frac{2(-4+4) - 2(1-4+4)}{4}$$

$$= \frac{-2}{4} = -\frac{1}{2}$$

$$\therefore \Delta f = -1\,\Delta x - \frac{1}{2}\Delta y$$

$$\left.\begin{array}{l} \Delta x = .01 \\ \Delta y = .03 \end{array}\right\} = -.01 - .015 = -.025$$

If $z = f(x,y)$, we sometimes write

$$dz = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$

Ex. If $f(x,y) = x^2 + 3xy - y^2$, find $\underline{\underline{dz}}$

$f_x = 2x + 3y \qquad f_y = 3x - 2y$

$\therefore\ df = (2x+3y)\,dx + (3x-2y)\,dy$

Put $x = 3$, $y = 2$,

and $\Delta x = .1$ and $\Delta y = .2$

$$(2x+3y) = 6+6 = \cancel{6}\ 12$$

$$(3x-2y) = 9-4 = \underline{\underline{5}}$$

$$\therefore \Delta z = \underset{12}{\cancel{0}}\cdot(.1) + 5(.2) = \underline{\underline{\cancel{1.0}}}\ 2.2$$

For functions of 3 variables:

$$\begin{aligned} f(x,y,z) = f(a,b,c) &+ f_x(a,b,c)(x-a) \\ &+ f_y(a,b,c)(y-b) \\ &+ f_z(a,b,c)(z-c) \end{aligned}$$

Ex. Find the first partial derivatives of

$$f(x,y,z) = xz - 5x^2y^3z^4$$

$$f_x = z - 10xy^3z^4$$

$$f_y = -15x^2y^2z^4$$

$$f_z = x - 20x^2y^3z^3$$

Ex. Find all second derivatives

of $f(x,y) = x^3y^5 + 2x^4y$

$f_x = 3x^2y^5 + 8x^3y$

$f_y = 5x^3y^4 + 2x^4$

$f_{xx} = 6xy^5 + 24x^2y$

$f_{xy} = 15x^2y^4 + 8x^3$

$f_{yy} = 20x^3y^3$

$f_{xy} = f_{yx} = 15x^2y^4 + 8x^3$

Ex. Does $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^4+y^4}$ exist

$(f(0,0)=0)$

On x-axis, $y=0 \to f(x,0)=0$ all x

On y-axis $x=0 \to f(0,y)=0$ all y

On $y=x$, $f(x,x)=\dfrac{x^2}{2x^4}=\dfrac{1}{2x^2}\to\infty$

as $x\to 0$.

$\therefore$ limit does NOT exist.