14.5

Chain Rule:

Find $\frac{\partial u}{\partial s}$ when $s=2$, $t=1$

$$u = x^2y + y^2 \qquad x = st + t^2 \qquad y = s^2 - t^3$$

$$\frac{\partial u}{\partial x} = 2xy \qquad \frac{\partial u}{\partial y} = x^2 + 2y$$

$$\frac{\partial x}{\partial s} = t \qquad \frac{\partial y}{\partial s} = 2s$$

$$\frac{\partial x}{\partial t} = s + 2t \qquad \frac{\partial y}{\partial t} = -3t^2$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= 2xy \cdot t + (x^2 + 2y)(-\cancel{3t^2})\, 2s$$

$\therefore x = 2+1 = 3 \quad , \quad y = 4-1 = 3$

$= 2 \cdot 9 \cdot 1 + (9+6)(-3)$

$= 18 - 45 = -27$

Similarly

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex Suppose $u = xy^2 - x^3 + y^2$

and $x = 3s - \frac{1}{s}$, $y = s^2 + 3s$

Find $\frac{du}{ds}$ for any s :

$$\frac{du}{ds} = \frac{\partial u}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial u}{\partial y} \frac{dy}{ds}$$

$$\frac{\partial u}{\partial x} = y^2 - 3x^2 \qquad \frac{\partial u}{\partial y} = 2xy + 2y$$

$$\frac{dx}{ds} = 3 + \frac{1}{s^2} \qquad \frac{dy}{ds} = 2s + 3$$

$$\frac{du}{ds} = (y^2 - 3x^2)(3 - \frac{1}{s^2})$$

$$+ (2xy + 2y)(2s + 3)$$

Now substitution

$x = 3s - \frac{1}{s}$ $\quad y = s^2 + 3s$

Ex. Pressure P (in kilopascals)

and Volume V (in liters)

and Temperature T (in kelvins)

are related by $PV = 8.31\,T$.

How fast is pressure changing

when the temperature is 300K

and increasing at .1 K/s and

the volume is 100 L and

increasing at .2 L/s.

$P = 8.31 \; T/V$

$$\therefore \quad \frac{dP}{dt} = \frac{\partial P}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial P}{\partial V} \cdot \frac{dV}{dt}$$

$$= \frac{8.31}{100} \cdot (0.1) - \frac{8.31(300)}{100^2} (0.2)$$

$$= -.042 \text{ kPa/s}$$

Write out the Chain Rule for the case when $w = f(x, y, z, t)$ and $x = x(u,v)$ $y = y(u,v)$, and $z = z(u,v)$ and $t = t(u,v)$.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$+ \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

AND

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$+ \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}$$

I made a mistake on Wed.

Suppose we have 2 resistors in parallel. The total resistance is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

Suppose the error is $\leq .005\,R_1$ and $\leq .005\,R_2$ when $R_1 = 20$ and $R_2 = 40$.

$$\frac{1}{R} = \frac{1}{20} + \frac{1}{40} = \frac{3}{40}$$

$$\rightarrow R = \frac{40}{3}.$$

Write out differential in all 3

variables

$$-\frac{1}{R^2}\,dR = -\frac{1}{R_1^2}\,dR_1 - \frac{1}{R_2^2}\,dR_2$$

$$\rightarrow dR = \frac{R^2}{R_1^2}\,dR_1 + \frac{R^2}{R_2^2}\,dR_2$$

Plug in $R^2 = \frac{1600}{9}$, $R_1^2 = 400$

$R_2^2 = 1600$

and $dR_1 \approx .005 \cdot 20$

and $dR_2 \approx .005 \cdot 40$

Implicit Differentiation

Ex. $y(x)$ is determined by

$$x^2 + (y(x))^2 = 1$$

Because we have square roots we can solve for $y(x)$ and compute $y'(x)$:

$$y(x)^2 = 1 - x^2$$

$$\rightarrow y(x) = \pm\sqrt{1-x^2}$$

and

$$y'(x) = \pm \frac{x}{\sqrt{1-x^2}}$$

Can we solve for $y(x)$ if

$$x^4 + y^6 + y^2 = 1$$

NO!!

But there is a way of finding $y'(x)$

Suppose $y(x)$ satisfies

$$F(x, y(x)) = 0.$$

Differentiate with respect to x

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

or more precisely

$$\frac{\partial F}{\partial x}(x, y(x)) + \frac{\partial F}{\partial y}(x, y(x)) \cdot \frac{dy}{dx} = 0$$

Now solve for $y'(x) = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}(x, y(x))}{\frac{\partial F}{\partial y}(x, y(x))}$$

or more briefly:

$$\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Ex. Find $\frac{dy}{dx}$ if $y\cos x - x^2 - y^2 = 0$

$$\frac{\partial F}{\partial x} = -y\sin x - 2x$$

$$\frac{\partial F}{\partial y} = \cos x - 2y$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(-y\sin x - 2x)}{\cos x - 2y}$$

$$\frac{dy}{dx} = \frac{y\sin x + 2x}{\cos x - 2y}. \qquad (1)$$

But what is y ?

In this case $x = 0$, $y = 1$ lies

on the curve.

$1 \cdot 1 - 0 - 1 = 0$ ✓

$$\frac{\partial x}{\partial s} = re^{t} \quad \frac{\partial y}{\partial s} = 2rse^{-t} \quad \frac{\partial z}{\partial s} = r^2 \sin t$$

$$\frac{\partial u}{\partial s} = (4x^3y)re^{t} + (x^4 + 2yz^3)(2rse^{-t})$$

$$+ (3y^2z^2)(r^2 \sin t)$$

Now solve when $r = 2,\ s = 1,\ t = 0$

$x = 2,\quad y = 2 \quad z = 0$

Note that f has a double point at i. Hence

$$\operatorname{Res}_i f(z) = \lim_{z \to i} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z-i)^n f(z).$$

or since $n=2$

$$\operatorname{Res}_i f(z) = \lim_{z \to i} \frac{d}{dz} \left((z-i)^2 \cdot f(z) \right)$$

$$= \lim_{z \to i} \frac{d}{dz} \left(\frac{1}{(z+i)^2} \right)$$

$$= -2 \left(\frac{1}{2i} \right)^3 = \underline{\underline{\frac{-i}{4}}}$$

deg P $\leq$ degree $Q - 2$,

with $R(x) \neq 0$ on the real line,

$$\int_{\gamma_R} R(z) = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k} R(z),$$

where z_k, $k=1,\dots,N$ is the set all roots of $Q(z)$ (i.e., the set of all poles in the upper plane) then by letting $R \to \infty$, we get

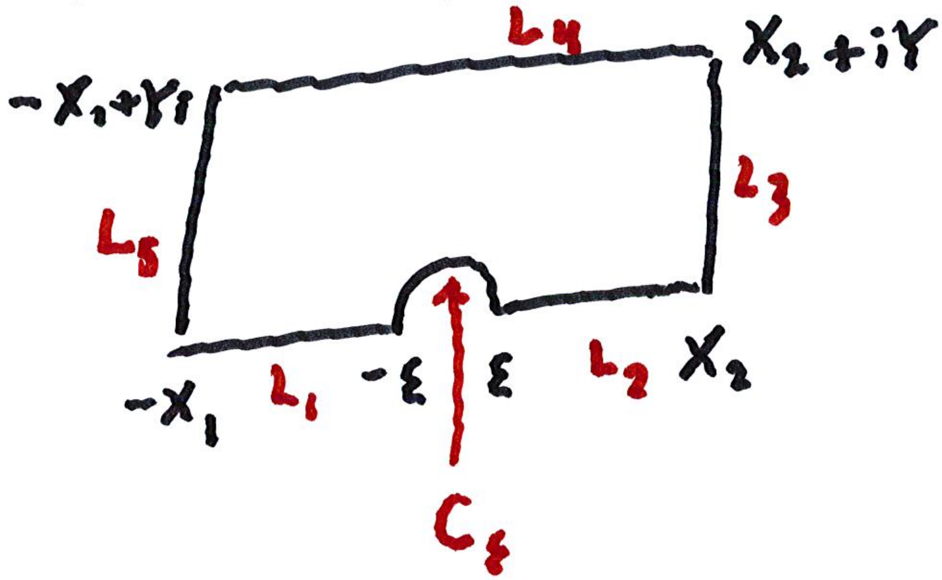
$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z_k} R(z).$$

Now we consider integrals where $\sin x$ or $\cos x$ appear in the numerator.

Find $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$.

Ex. Compute $\int_0^\infty \frac{\sin x}{x}\,dx$.

We use the path



We study $\int_\gamma \frac{e^{iz}}{z}\,dz$ on the above path.

As X_1 and $X_2 \to \infty$ and $\varepsilon \to 0$.

$$\int_{L_1} + \int_{L_2} \longrightarrow \int_{-\infty}^{\infty}$$

$$\left| \int_{X_2}^{X_2+iY} e^{i(X_2+iy)} \cdot \frac{1}{z} \, dy \right|$$

$$\le \int_0^Y e^{-y} \cdot \frac{1}{X_2} \, dy \le \frac{1}{X_2} \to 0$$

as $X_2 \to \infty$

$$\therefore \left| \int_{L_3} \frac{e^{iz}}{z} \right| \to 0 \qquad \int_{L_5} \to$$

$$\left| \int_{L_4} \frac{e^{iz}}{z} dz \right| \le \frac{e^{-Y}}{Y} (X_2 + X_1) \to 0$$

for fixed X_1 and X_2 as $Y \to \infty$

$$\therefore \left| \int_{L_4} \frac{e^{iz}}{z} dz \right| \to 0.$$

Note that

$$\frac{e^{iz}}{z} = \frac{1}{z} \left[1 + \frac{(iz)}{1!} + \frac{(iz)^2}{2!} + \cdots \right]$$

Finally, we estimate

$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz \quad , \text{ by using } z(t) = \varepsilon e^{it}.$$

for $0 \leq t \leq \pi$

we get that the sum of all but the first term can be estimated by $\displaystyle\sum_{k=1}^{\infty} \frac{\varepsilon^{k-1}}{k!} \cdot \pi\varepsilon$

$\leq \pi(e^{\varepsilon} - 1)$, which $\to 0$ as $\varepsilon \to 0$

On the other hand ~~the~~,

$$\int_0^{\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}}\, dt = \pi i.$$

Since the integral of

$\frac{e^{iz}}{z}$ should be taken in

the clockwise direction, we

get that $+\int_{C_\varepsilon} e^{iz}\, \frac{e^{iz}}{z}\, dz$

$= \cancel{-\pi i} + O(\varepsilon).$

$+\pi i$

or $1-y=0 \Rightarrow$

$\rightarrow y=1 \Rightarrow x=\sqrt{2}$

$\therefore (0,0)$ is a crit pt.

and $(1, \pm\sqrt{2})$

Note that $(1, \pm\sqrt{2})$ lie

outside of D.

$f_{xx} = 2-2y \quad f_{yy} = 2$

$f_{xy} = -2x$