14.6 Directional Derivatives

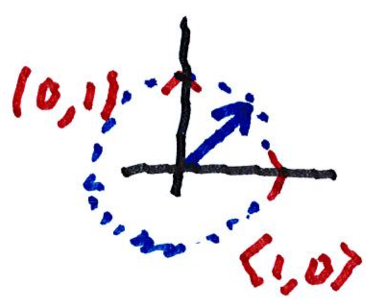
$$\lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \text{rate of change}$$

in the $\langle 1,0 \rangle$ direction

and

$$\lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} = \text{rate of change in the}$$

$\langle 0,1 \rangle$ direction



Note that $\langle 1,0 \rangle$ and $\langle 0,1 \rangle$ are unit vectors.

Now let $\vec{u}$ be any unit vector

We define

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Ex. Suppose $x = x_0 + ha$ and $y = y_0 + hb$

We want to figure out the

range of change as $h \to 0$ when $h = 0$

$$\frac{\partial f}{\partial h} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh}$$

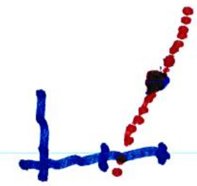
$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0) b$$

or:

$$D_{\vec{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b$$

This is called the directional derivative of f at (x, y) in the direction of a unit vector $\vec{v}$

Ex. Let $f(x,y) = x^2y - y^2 - x^3$

Compute $D_{\vec{v}} f(x_0, y_0)$ 

at $x_0 = 1$, $y_0 = 2$ and $\vec{v} = \frac{3}{\sqrt{10}}\vec{i} + \frac{1}{\sqrt{10}}\vec{j}$

$$\frac{\partial f}{\partial x} = 2xy - 3x^2 = 4 - 3 = 1$$

$$\frac{\partial f}{\partial y} = x^2 - 2y = -3$$

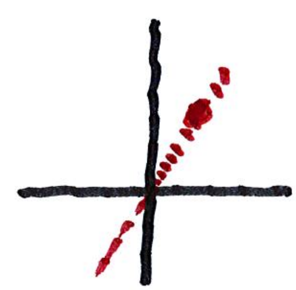
$$D_{\vec{v}} f = 1 \cdot \frac{3}{\sqrt{10}} - 3 \cdot \frac{1}{\sqrt{10}} = 0$$

Ex. Find the directional derivative $D_{\vec{u}} f(x,y)$ if

$f(x,y) = x^2 - xy^3 + y^2$ and

$\vec{u}$ is the unit vector pointing in the direction with angle $\theta = \frac{\pi}{3}$

and $(x_0, y_0) = (2,1)$.



$$f_x = 2x - y^3 = 4 - 1 = 3$$

and

$$f_y = -3xy^2 + 2y = -6 + 2 = -4$$

Since $\vec{u} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$,

$$D_{\vec{u}} f(2,1) = 3 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2} - 2\sqrt{3}$$

Note that we can write

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$$

$$= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$$

We call this

$$\nabla f_{(x,y)} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Hence $D_{\vec{u}} f(x,y) = \underline{\underline{\nabla f(x,y) \cdot \vec{u}}}$

Ex. Find the directional derivative of the function $f(x,y) = x^2y^3 - y^2$ at the point $(3, 2)$ in the direction of $\vec{u} = \dfrac{2\vec{i} + 3\vec{j}}{\sqrt{13}}$

$$\frac{\partial f}{\partial x} = 2xy^3 = 6 \cdot 8 = 48$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 2y = 108 - 4 = 104$$

$\therefore \nabla f_{(3,2)} = 48\vec{i} + 104\vec{j}$

$\Rightarrow D_{\vec{u}} f = 48 \cdot \frac{2}{\sqrt{13}} + \frac{104 \cdot 3}{\sqrt{13}}$

$= \frac{408}{\sqrt{13}}$

Functions of 3 variables:

Given a function $f(x, y, z)$

and a unit vector $\vec{u} = \langle a, b, c \rangle$

at (x_0, y_0, z_0), then we define

$D f_{\vec{u}}(x_0, y_0, z_0)$

$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

We can compute $D_{\vec{u}} f(x, y, z)$

$$= \frac{\partial f}{\partial x}(x, y, z)\, a + \frac{\partial f}{\partial y}(x, y, z)\, b + \frac{\partial f}{\partial z}(x, y, z)\, c$$

We can define the directional derivative by

$D_{\vec{v}} f = \nabla f \cdot \vec{v}$

The quantity $\nabla f \cdot \vec{v}$

depends on which vector we choose.

Recall the formula

$$\nabla f \cdot \vec{v} = |\nabla f|\,|\vec{v}| \cos\theta$$

$$= |\nabla f| \cos\theta$$

We get the largest value when $\cos\theta = 1$, i.e., when $\vec{u}$ points in the same direction as ∇f.

But remember, $\vec{u}$ must be a unit vector. i.e, when

$$\vec{u} = \frac{\nabla f}{|\nabla f|}$$

Tangent Planes to level surfaces.

Note that if we travel in the unit direction $\vec{u}$, then the rate of change is $= 0$ if the motion vector $\vec{u}$ satisfies $\nabla f \cdot \vec{u} = 0$.

Than $\nabla f \cdot \vec{u}$

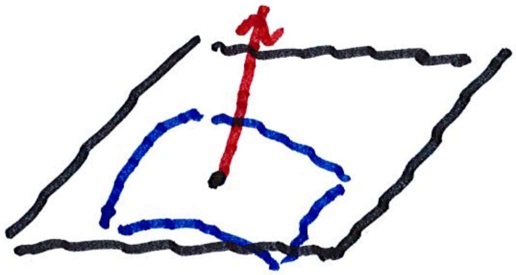
$$= \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|}$$

$$= |\nabla f|$$

In sum, $\nabla f \cdot \vec{u}$ is maximized

when $\vec{v} = \dfrac{\nabla f}{|\nabla f|}$ and the largest

value is $= |\nabla f|$



We can define the tangent plane by

$$
\begin{aligned}
& F_x(x_0, y_0, z_0)(x - x_0) \qquad (1) \\
& + F_y(x_0, y_0, z_0)(y - y_0) \\
& + F_z(x_0, y_0, z_0)(z - z_0) = 0 .
\end{aligned}
$$

Ex. Find the tangent plane

to the level surface

$$f(x,y,z) = k$$

through (x_0, y_0, z) by using (1).

Ex. Find the equation of the

surface $\underbrace{x^2 - y^2 + 2z^2 + 5}_{F(x,y,z)} = 0.$

through $(1,2,-1)$

$$\nabla f = 2x\vec{i} - 3y^2\vec{j} = 2\vec{i} - 12\vec{j}$$

The line that is $\perp$ to the gradient is $(6, 2)$

$\therefore$ the slope is $\frac{2}{6} = \frac{1}{3}$

$\therefore y - 2 = \frac{1}{3}(x - 1)$