Maxima and Minima

Ex. (from 14.6) and 14.7 (beginning)

If $f(x,y) = x e^y - x^2$

Find rate of change at $(2,0)$

in the direction from $P(2,0)$

to $Q(1,2)$.

$$\nabla f = (e^y - 2x)\vec{i} + (xe^y)\vec{j}$$

$$= (1-4)\vec{i} + 2\vec{j}$$

$$= -3\vec{i} + 2\vec{j}$$

$$Q - P = (1, 2) - (2, 0)$$

$$= -\vec{i} + 2\vec{j}$$

$$\therefore \vec{u} = \frac{-1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$$

$$\nabla f \cdot \vec{u} = \langle -3, 2 \rangle \cdot \left\langle \frac{-1}{\sqrt{5}}, \frac{+2}{\sqrt{5}} \right\rangle$$

$$= \frac{+3}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{+7}{\sqrt{5}}.$$

Direction of Max. Change

$$\vec{u} = \underline{\frac{\nabla f}{|\nabla f|}} = \frac{\langle -3, 2\rangle}{\sqrt{3^2+2^2}} = \left\langle \frac{-3}{\sqrt{13}}, + \frac{2}{\sqrt{13}} \right\rangle$$

Max. Rate of change

$$= \sqrt{(-3)^2 + 2^2} = \sqrt{13}$$

Ex. Find eq'n of tangent

plane of $x^2 + 4y^2 + 4z^2 = 9$

at $(1, -1, 1)$.

$$\nabla f \cdot \langle x - x_0 ,\ y - y_0 ,\ z - z_0 \rangle = 0$$

$$\nabla f = \langle 2x,\ 8y,\ 8z \rangle$$

$$= \langle 2,\ -8,\ 8 \rangle$$ at $(1,-1,1)$

$$\underline{2(x-1) - 8(y+1) + 8(z-1) = 0}$$

Find the equation of the

normal line through (x_0, y_0, z_0)

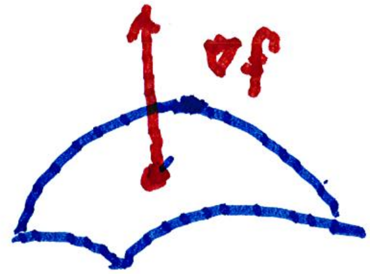
and that is $\perp$ to tangent plane

$$\nabla f = \langle f_x, f_y, f_z \rangle \text{ at } (x_0, y_0, z_0)$$

$$x - x_0 = t f_x$$

$$y - y_0 = t f_y$$

$$z - z_0 = t f_z$$



Def'n: A fcn. $f(x,y)$ of two variables has a local maximum at (a,b) if $f(x,y) \le f(a,b)$ if (x,y) is close to (a,b).

A fcn. $f(x,y)$ has a local minimum at (a,b) if

$f(x,y) \ge f(a,b)$ if (x,y) is close to (a,b)

If the inequalities hold

for all x, y, then we say

f has an absolute maximum

(or absolute minimum) at (a,b)

Thm. If f has a local maximum

(or local minimum), then

$$f_x(a,b)=0 \quad \text{and} \quad f_y(a,b)=0.$$

(a,b) is called a critical point

Ex. Find all critical points of

$$f(x,y) = x^2 - 2y^2 - 4x + 8y - 1$$

$$f_x = 2x - 4 = 0 \rightarrow x = 2$$

$$f_y = -4y + 8 = 0 \rightarrow y = 2$$

$(2,2)$ is the <u>only</u> critical point

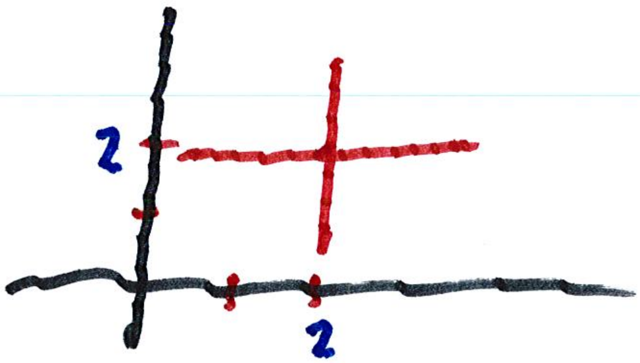
Complete the square:

$$x^2 - 4x + 2(y^2 - 4y) = 1 \cancel{+4}$$

$$(x-2)^2 + 2(y-2)^2 = \text{[illegible]}$$

$$= 13.$$

$(x-2)^2 \geq 0 \quad (y-2)^2 \geq 0$



$\therefore$ $f(x,y)$ has an absolute minimum

at $(2,2)$.

Ex Find the extreme values

of $f(x,y) = x^2 - y^2$

$f_x = 2x \quad f_y = 2y$

$\rightarrow$ $(0,0)$ is a critical point.

But set $x = t$, $y = 0$

$f(t,0) = t^2$. which increases

$f(0,t) = -t^2$ $\therefore$ f decreases

f has a saddle point

(b) If $D > 0$ and $f_{xx}(a,b) < 0$

then f has a local maximum

(c) If $D < 0$, then $f(a,b)$ is

not a local minimum or

a local maximum.

(In fact f has a saddle point.

If $D = 0$, the test ~~has~~ gives no information.

Ex. Find the local maximum and local minimum values for

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f_x = 4x^3 - 4y = 0 \rightarrow y = x^3 \quad (1)$$

$$f_y = 4y^3 - 4x = 0 \rightarrow x = y^3. \quad (2)$$

Plug (1) into (2)

$\rightarrow x = x^9 \longrightarrow x = 0$ or

$1 = x^8 \rightarrow (1-x^4)(1+x^4)$

$\rightarrow (1-x^2)(1+x^2)(1+x^4)$

$\therefore x = 0,\ 1$ or -1

$\downarrow \quad \downarrow \quad \downarrow$

$(0,0) \quad (1,1) \quad (-1,-1)$

Now calculate

$D_{xx}f = 12x^2\ ,\ D_{yy} = 12y^2$

$f_{xy} = -4 + 12x^2$

$\therefore$ for $(0,0)$ $D_{xy} = -4$

$\rightarrow$ $(0,0)$ is a saddle point

$f_{xx} = 12x^2$ $f_{xy} = -4$

$f_{xy} = -4$ $f_{yy} = 12y^2$

$\therefore$ For $(1,1)$ $D = 12^2 - 16 > 0$

and $f_{xx} > 0$ $\Rightarrow$ $(1,1)$ is a local min

For $(-1,-1)$ ~~D...~~ $D = 12\delta^2 -$

$\rightarrow$ is local min

$- (f_{xy})^2 = -16 + 144x^2y^2$. $D < 0$ saddle pt.

Ex. 5 Find the shortest distance from $(1, 0, -2)$ to the plane

$$x + 2y + z = 4.$$

$$d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}.$$

We can minimize the square by minimizing the square

$$d^2 = f(x,y) = (x-1)^2 + y^2 + (\underbrace{4 - x - 2y}_{z} + 2)^2$$

$$f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$f_x = 2(x-1) + 2(6-x-2y)(-1)$$

$$=$$ [illegible]

$$f_x = 4x + 4y - 14 = 0.$$

$$f_y = 2y + 2(6-x-2y)(-2)$$

$$=$$ [illegible] $$10y + 4x - 24 = 0$$

$$\therefore \quad 4x + 4y = 14$$

$$4x + 10y = 24$$

$$\rightarrow \quad 6y = 10 \rightarrow y = \cancel{\tfrac{10}{3}} \; \frac{10}{6} = \frac{5}{3}$$

$$\therefore \quad 4x = 14 - 4y = 14 - \frac{20}{3} = \frac{22}{3}$$

$$\rightarrow x = -\frac{22}{12} = -\frac{11}{6}$$

$\therefore$ Only crit. point is $\left(\frac{11}{6}, \frac{5}{3}\right)$

Intuitively, this is a local minimum.

To find an absolute minimum

on a H closed bounded set:

1. Find the values at the critical points.

2. Find the extreme values of f on the boundary of D.

3. Compare the values from (1) and (2)

33 Find the absolute maximum and minimum points of

$$f(x,y) = x^4 + y^4 - 4xy + 2$$

$$D = \{ (x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2 \}$$

$f_x = 4x^3 - 4y = 0 \rightarrow y = x^3$

$f_y = 4y^3 - 4x \quad x = y^3$

$x = y^3 = (x^3)^3 \rightarrow x = 0$ or $1 = x^8$

$x = x^9 \rightarrow 1 = x^8$

If $x=0 \to y=0$

If $x=-1$

$\downarrow$

$y = (-1)^8 = 1$

If $x=1$

$\downarrow$

$y=1$

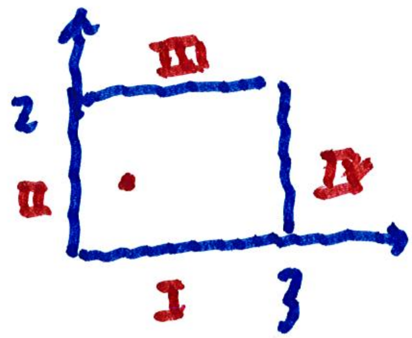
$\therefore$ 3 crit. points

$(0,0)$ $(1,1)$ $(-1,1)$

$\uparrow$ not in interior

$\downarrow$ in int. of D

$f(1,1) = 2-4+2 = 0$



$$f(x,0) = x^4 + 0 + 0 + 2 = x^4 + 2$$

I $\underline{f(3,0) = 83}$ $\quad f(0,0) = 2$

II $f(0,y) = y^4 + 2 \leq 18 \quad \underline{2^4 + 2 = 18}$

$y = 0 \to 2$

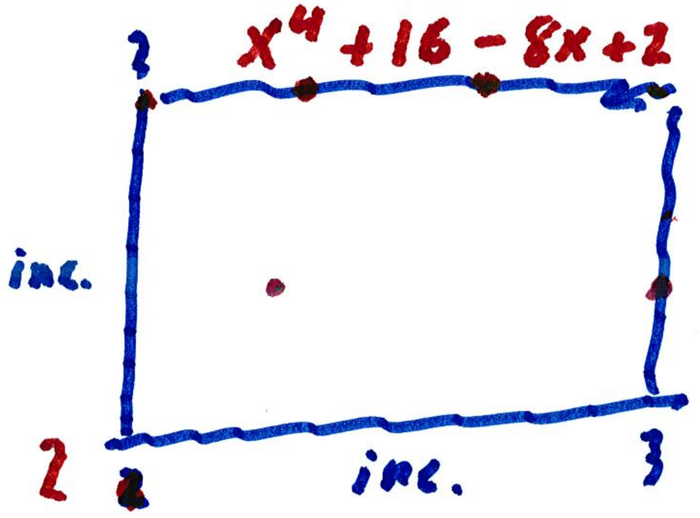
III $f(x,2) = x^4 + 16 - 8x + 2 \approx 87$

$-4xy$

Find max of $x^4 + 16 - 8x + 2$

$$y'(x) = 4x^3 - 8$$

$$\rightarrow x^3 = 2 \rightarrow x = \sqrt[3]{2}$$



$81 + y^4 - 12y + 2$

IV $4y^3 - 12 = 0$

$y^3 = 3$

It appears that $f(1,1)$ is

the minimum value:

$f(1,1) = 0$

#31. $f(x,y) = x^2 + y^2 - x^2y + 4$

$D = |x| \leq 1, \ |y| \leq 1$

$f_x = 2x - 2xy \qquad f_y = 2y - x^2 = 0$

$= 2x(1-y) = 0 \qquad \rightarrow y = \dfrac{x^2}{2}$

$x = \sqrt{2}$

$x = 0 \rightarrow y = 0 \quad (0,0) \quad (\sqrt{2}, \qquad x = \pm\sqrt{2}$

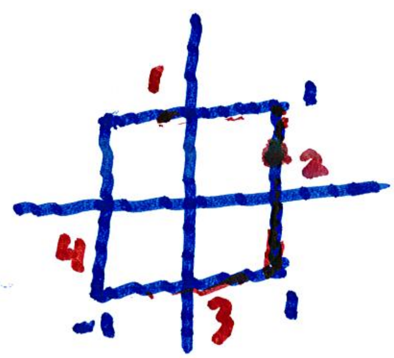
$(-\sqrt{2}, 1)$ $(\sqrt{2}, 1)$

For $(0,0)$

$$D = \begin{bmatrix} 2 & -2x \\ -2x & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$D = 4 - 0 \Rightarrow$ local min or a local max

$(0,0)$



Look at $(x,1)$ $-1 \leq x \leq 1$

1. $f(x,1) = x^2 + 1 - x^2 + 4 = 5$

$$f(1,y) = 1 + y^2 - y + 4$$
$$= y^2 - y + 3$$