14.8 (Lagrange Multipliers)

As the picture suggests, one should look at those points where both curves have the same tangent line.

This means that the gradient of $f(x_0, y_0)$ is a multiple of the gradient of g.

$\therefore \quad \nabla f_{(x,y)} = \lambda \nabla g_{(x,y)}$

and $g(x,y) = k$

This gives 3 equations.

If $\nabla f = \langle f_x , f_y \rangle$ and

$\nabla g = \langle g_x , g_y \rangle$.

or:

$$\boxed{\begin{array}{l} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \quad \text{and} \\ g(x,y) = k \end{array}}$$

Or

Ex. Find the maximum and minimum of $f(x,y) = x^2 - 2x + 2y^2$ on the ~~plane~~ line $4x - 6y = 1$

Thus, $g(x,y) = 4x - 6y = 1$

$\nabla f = \lambda \nabla g$, i.e.

$f_x = \lambda g_x$

$f_y = \lambda g_y$

$g(x,y) = k$

$\therefore \quad 2x - 2 = \lambda \cdot 4$

$4y = \lambda \cdot (-6)$

and $\quad 4x - 6y = 1$

$\rightarrow \quad x = 2\lambda + 1$

$y = -\frac{3}{2}\lambda$

Sub into g-equation

$4(2\lambda + 1) - 6\left(-\frac{3}{2}\lambda\right) = 1$

$\hookrightarrow \quad 26\lambda = -3$

Plug back into equations for

x and y

$$x = 2\left(\frac{-3}{26}\right) + 1 = \frac{10}{13}$$

$$y = -\frac{3}{2}\left(\frac{-3}{26}\right) = \frac{9}{52}$$

This works in any dimensions

Ex. Find the maximum any minimum of $f(x,y,z) = x + 2y + 4z$

on the surface $\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 1$

$$1 = 2\lambda x$$

$$2 = 2\lambda y$$

$$4 = 2\lambda z \quad , \quad x^2+y^2+z^2 = 1$$

Solve for x, y, and z

in terms of λ:

$$x = \frac{1}{2\lambda}$$

$$y = \frac{1}{\lambda}$$

$$z = \frac{2}{\lambda}$$, and $x^2 + y^2 + z^2 = 1$

$$\therefore \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 = 1$$

$$\frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{16}{4\lambda^2} = 1$$

$$\frac{21}{4\lambda^2} = 1 \rightarrow 4\lambda^2 = 21$$

$$\rightarrow \lambda = \frac{\pm\sqrt{21}}{2}$$

$$\lambda = +\frac{\sqrt{21}}{2} \rightarrow x = \frac{1}{2} \cdot \frac{2}{\sqrt{21}} = \frac{1}{\sqrt{21}}$$

$$y = \frac{2}{\sqrt{21}}$$

$$z = \frac{4}{\sqrt{21}}$$

$$\left(\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4}{\sqrt{21}} \right)$$

If $\lambda = \frac{-\sqrt{21}}{2}$, then

$$\left(\frac{-1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}, \frac{-4}{\sqrt{21}} \right)$$ Plug into

Ex. Find the min and max of

$\cancel{f(x,y,z)=}$ $f(x,y) = e^{xy}$ on the

curve $x^3 + y^3 = 16$

$f(x,y)$

$g(x,y)$

$$\nabla f = \lambda \nabla g, \quad g(x,y) = k$$

$$\nabla(e^{xy}) = \lambda(\nabla g) \qquad \nabla g = \langle 3x^2, 3y^2 \rangle$$

$$f_x = ye^{xy} = \lambda \cdot 3x^2 \qquad f_y = xe^{xy} = \lambda \cdot 3y^2$$

$$\lambda = \frac{ye^{xy}}{3x^2} \qquad \lambda = \frac{xe^{xy}}{3y^2}$$

$$\therefore \quad \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$$

$$\rightarrow \quad 3y^3 = 3x^3$$

$$y^3 = x^3 \rightarrow (y-x)\underbrace{(y^2+yx+x^2)}_{>0}$$

$\therefore y = x$ sub into g-eq'n:

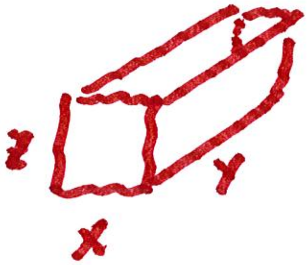
$$2\cancel{x}^3 = 16$$

$\rightarrow x^3 = 8 \rightarrow x = 2$

$\therefore y = x \rightarrow y = 2$

$\therefore$ crit. pt is $(2, 2)$

on $x^3 + y^3 = 16$



Total area $= xy + 2xz + 2yz$

$= 12$

Find x, y, z that maximize $V = \underbrace{xyz}_{f(x,y,z)}$

$g \quad xy + 2xz + 2yz = 12 \qquad \nabla g = \langle y+2z,\ x+2z,\ 2x+2y \rangle$

$yz = V_x = \lambda(y+2z)$

$xz = V_y = \lambda(x+2z)$

$xy = V_z = \lambda(y+2z)$

Multiply by x, y, z resp.

$$xyz = \lambda(xy + 2xz)$$

$$xyz = \lambda(2yz + xy)$$

$$xyz = \lambda(2xz + 2yz)$$

$\lambda \neq 0$, ~~because~~ because that would imply $yz = xz = xy = 0$ which would contradict g-equation.

So we get

$$2xz + xy = 2yz + xy$$

$$\rightarrow \quad 2xz = 2yz \quad \rightarrow \quad \underline{\underline{y = x}}$$

But then, we can conclude

from that $y = 2z$.

Hence we have $x = y = 2z$

$$\Rightarrow \quad 4x^2 + 4y^2 + 4z^2 = 12$$