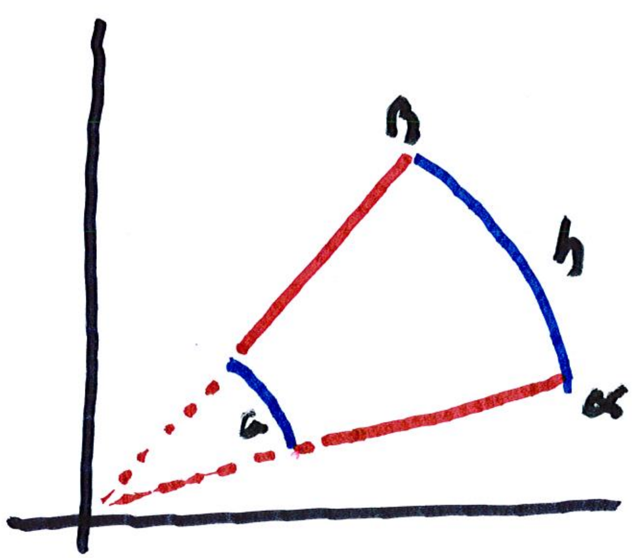
15.4 Double Integrals in Polar Coordinates.

Given a "polar rectangle".

how do we write it as a sum

of many small "polar rectangles



$\alpha < \theta < \beta$

$a \leq r \leq b$

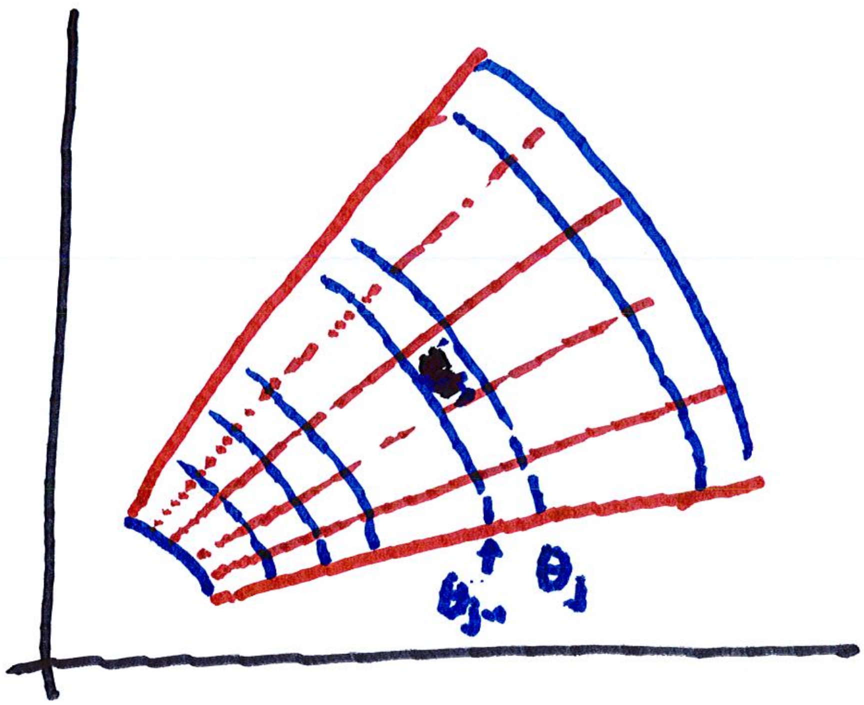
Subdivide

$$a = r_0 < r_1 < r_2 \ldots < r_{j-1} < r_j < \ldots < r_m = b$$

where $\quad r_j - r_{j-1} = \Delta r = \dfrac{b-a}{m}$

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_{k-1} < \theta_k \cdots < \theta_n = \beta$$

where $\quad \theta_k - \theta_{k-1} = \Delta\theta = \dfrac{\beta - \alpha}{n}$



$$\Delta A = \pi\left(r_j^2 - r_{j-1}^2\right) \cdot \frac{\Delta\theta}{2\pi}$$

$$= \pi\left(r_j + r_{j-1}\right)\left(r_j - r_{j-1}\right) \cdot \frac{\Delta\theta}{2\pi}$$

If we assume that $r_{j-1} \approx r_j$

then $\Delta A \approx \pi r_j \Delta r \cdot \dfrac{\Delta\theta}{\pi}$

or $\Delta A \approx r_j \Delta r \Delta\theta$

Now imagine that the height

of the box above the rectangle

is

$$f(x,y) = f(r_j \cos\theta_k , r_j \sin\theta_k)$$

then the volume of the solid

region is

$$V \approx \sum_{j=1}^{m} f(r_j \cos\theta_k , r_j \sin\theta_k)\, r_j \,\Delta r \,\Delta\theta$$

If we let $m \to \infty$ and $n \to \infty$,

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$$

Compute the integral

$$\iint_R (x^2+y)\,dA,$$ where R is the region in the first quadrant bounded by $x^2+y^2=1$ and $x^2+y^2=4$

$$I_1 = \int_0^{\frac{\pi}{2}} \int_1^2 r^3 \cos^2\theta \, dr \, d\theta \, .$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_1^2 \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta \left(4 - \frac{1}{4}\right) d\theta$$

$$= \frac{15}{4} \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{15}{8} \cdot \frac{\pi}{2} + \underbrace{\frac{15}{16} \sin 2\theta \Big|_0^{\frac{\pi}{2}}}_{=0}$$

$$\int_0^{\frac{\pi}{2}} \int_1^2 r^2 \sin\theta \, dr \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin\theta \cdot \frac{r^3}{3} \Big|_1^2$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta \cdot \frac{7}{3} \, d\theta$$

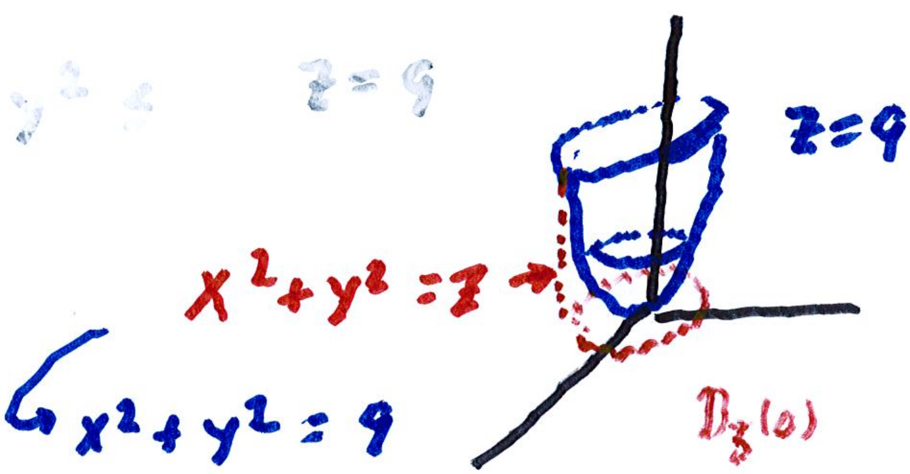
$$= \frac{7}{3} \left(-\cos\theta\right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{7}{3} \left(\cos 0\right) = \frac{7}{3}$$

$$\frac{15\pi}{16} + \frac{7}{3}$$

$$\therefore \int_R = \frac{15\pi}{16}$$

Ex. Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 9$



$$\text{Vol} = \int_0^{2\pi} \int_0^3 (9 - r^2)\, r \, dr \, d\theta$$

$$= 2\pi \int_0^3 9r - r^3 \, dr$$

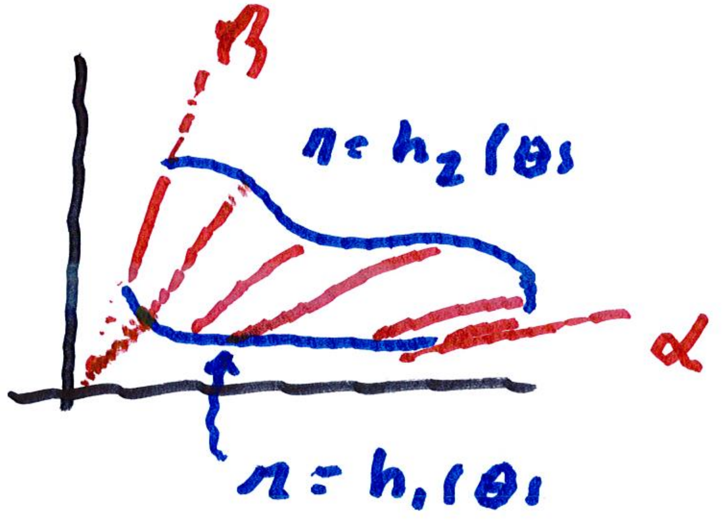
$$= 2\pi \left(\frac{9r^2}{2} - \frac{r^4}{4} \right) \Bigg|_0^3$$

$$= 2\pi \left(\frac{9 \cdot 9}{2} - \frac{81}{4} \right)$$

$$= \frac{81}{4} \cdot 2\pi = \frac{81\pi}{2}$$

Now suppose D is bounded by

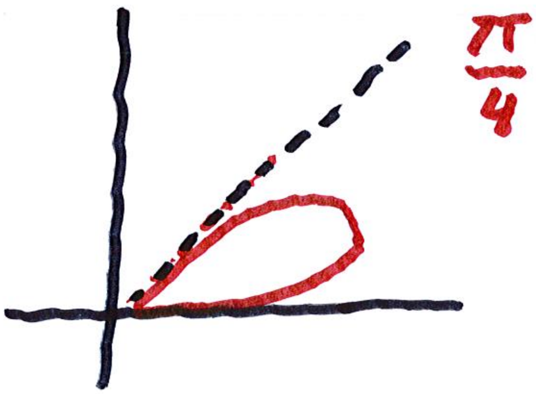
$$D: \{(r,\theta);\ \alpha \le \theta \le \beta,\ ,$$
$$h_1(\theta) \le r \le h_2(\theta)$$



$$\iint_R f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta$$

Use a double integral to find the area in one loop of $r = \sin 4\theta$.

$0 < \theta < \frac{\pi}{4}$



$$A = \int_0^{\frac{\pi}{4}} \int_0^{\sin 4\theta} 1 \cdot r \, dr \, d\theta$$

$$A = \int_0^{\pi/4} \frac{1}{2} \sin 4\theta \cdot r$$

$$= \int_0^{\pi/4} \int_0^{\sin 4\theta} r \, dr \, d\theta$$

$$= \int_0^{\pi/4} \cdot \frac{r^2}{2} \Big|_0^{\sin 4\theta} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^2 4\theta}{2} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 8\theta}{2 \cdot 2} \, d\theta$$

$$= \frac{\pi}{\cancel{8}\ 16} - \frac{\sin 8\theta}{\cancel{16}\ 32} \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{\cancel{8}\, 16}$$

Ex Find area inside $r = 1 + \cos\theta$

Let A = area of top half

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(h(\theta)\right)^2 \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\left(1 + \cos\theta\right)^2 \, d\theta$$

$$= \frac{\pi}{4} + \int_0^{\frac{\pi}{2}} \cos\theta + \frac{\cos^2\theta}{2} \, d\theta$$

$$\frac{\pi}{4} + \; L = \sin\theta \Big|_0^{\pi/2} + \int_0^{1+\cos}$$

$$= \frac{\pi}{4} + 1 + \int_0^{\pi/2} \frac{1+\cos 2\theta}{4}\, d\theta$$

$$= \frac{\pi}{4} + 1 + \frac{\pi}{8} + \underbrace{\left.\frac{\sin 2\theta}{8}\right|_0^{\pi/2}}_{=0}$$

$$= 1 + \frac{3\pi}{8}$$

$\therefore$ Area $= 2 + \dfrac{3\pi}{4}$

Find the vol. of the region

above the cone $z = \sqrt{x^2+y^2}$

and below $z = \sqrt{1-x^2-y^2}$

$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-r^2} - r\right) r \, dr$$