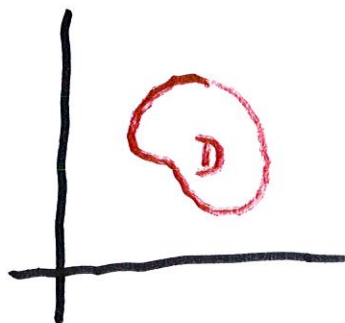


15.5 Density and Mass

Consider a thin plate in the shape of

a domain D .



The density of

D may vary

as a function of (x, y) . We define

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A} \rightarrow \underline{\Delta m \approx \rho(x, y) \Delta A}$$

In the usual way, we can

approximate D by a union of
rectangles $R_{i,j}$, each of area

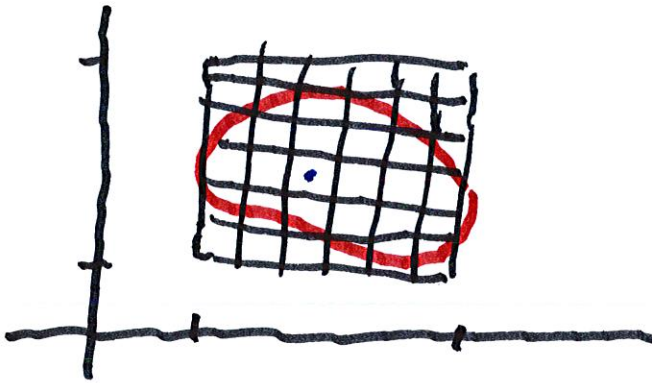
$$\Delta A = \Delta x \Delta y. \text{ The mass of}$$

the rectangle R_{ij} is approx.

$$= \rho(x_i, y_j) \Delta x \Delta y.$$

\therefore The total mass is approx

$$m \approx \sum_{i,j} \rho(x_i, y_j) \Delta x \Delta y$$



$$\text{So } m = \iint_D \rho(x, y) dA$$

Similarly, if $\sigma(x, y) =$ charge density

at (x, y) , then the total charge is

$$Q = \iint_D \sigma(x, y) dA$$

Moments and Center of Mass

In Chapter 8, we defined

$$\begin{aligned} \text{Moment of } D \\ \text{about } y\text{-axis} &= \sum_{i=1}^m \sum_{j=1}^n x_{ij} \rho(x_i, y_j) dA \end{aligned}$$

$$\begin{aligned} \rightarrow \iint_D x \rho(x, y) dA \\ = M_y \end{aligned}$$

and

$$\begin{aligned} \text{Mom. of } D \\ \text{about } x\text{-axis} &= \iint_D y \rho(x, y) dA \\ &= M_x \end{aligned}$$

5.

As before, the center of mass of

D is at (\bar{x}, \bar{y}) , where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m},$$

$$\text{where } m = \iint_D \rho(x, y) \, dA.$$

Ex. Find the center of mass of D if

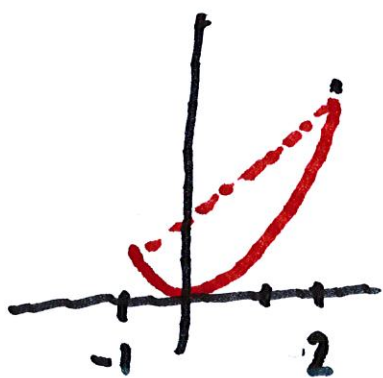
D is bounded by $y = x^2$ and $y = x + 2$

and $\rho(x, y) = y^2$.

$y = x^2$ and $y = x + 2$ intersect

where $x^2 = x + 2$ or $x^2 - x - 2 = 0$

$\rightarrow (x-2)(x+1) = 0 \rightarrow x = -1$ or $x = 2$



$$\underline{\underline{\text{Mass}}} = \iint_D y \, dA$$

$$= \int_{-1}^2 \int_{x^2}^{x+2} y \, dy \, dx$$

$$= \int_{-1}^2 \left. \frac{y^2}{2} \right|_{x^2}^{x+2} dx$$

$$= \int_{-1}^2 \left(\frac{(x+2)^2}{2} - \frac{x^4}{2} \right) dx$$

$m =$

$$\bar{x} = \frac{M_y}{m}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_{-1}^2 \int_{x^2}^{x+2} xy \, dy \, dx$$

$$= \int_{-1}^2 \left. \frac{xy^2}{2} \right|_{y=x^2}^{y=x+2} dx$$

$$= \int_{-1}^2 \left(\frac{x(x+2)^2}{2} - \frac{x^5}{2} \right) dx$$

$$M_y = \iint_D xy \, dA$$

$$= \int_{-1}^2 \int_{x^2}^{x+2} xy \, dy \, dx$$

$$= \int_{-1}^2 x \left. \frac{y^2}{2} \right|_{x^2}^{x+2} dx$$

$$M_y = \frac{1}{2} \int_{-1}^2 x(x+2)^2 - x^5 \, dx$$

$$\bar{y} = \frac{M_x}{m}, \quad \text{where}$$

$$M_x = \iint_D y^2 dA$$

$$= \int_{-1}^2 \left. \frac{y^3}{3} \right|_{x^2}^{x+2} dx$$

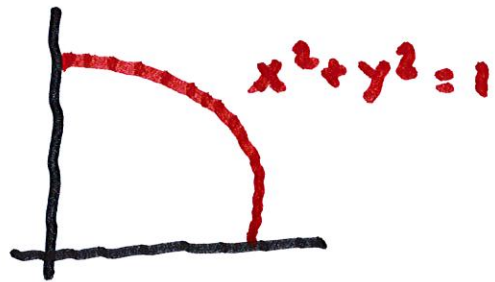
$$M_x = \frac{1}{3} \int_{-1}^2 (x+2)^3 - x^6 dx$$

Ex Find (\bar{x}, \bar{y}) if $D =$ region in

first ~~octant~~ ^{quadrant} bounded by

$x^2 + y^2 = 1$ and where the

density = y



$$\bar{x} = \frac{M_y}{m}$$

$$m = \iint_D y \, dA = \int_0^{\pi/2} \int_0^1 r \sin \theta \, r \, dr \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin \theta \, d\theta$$

$$= -\frac{1}{3} \cos \theta \Big|_0^{\pi/2} = \frac{1}{3}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_0^{\pi/2} \int_0^1 r \cos \theta \, r \sin \theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin \theta \cos \theta}{4} \, d\theta$$

$$= \frac{1}{4} \frac{\sin^2 \theta}{2} \Big|_0^{\frac{\pi}{2}} = \frac{1}{8}$$

$$\therefore \bar{x} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}$$

$$M_x = \iiint y \cdot y \, dA$$

$$= \int_0^{\pi/2} \int_0^1 (r \sin \theta)^2 r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2 \theta \, d\theta$$

$$= \frac{1}{8} \int_0^{\pi/2} (1 - \cos 2\theta) \, d\theta$$

$$= \left[\frac{\theta}{8} - \frac{\sin 2\theta}{16} \right]_0^{\pi/2}$$

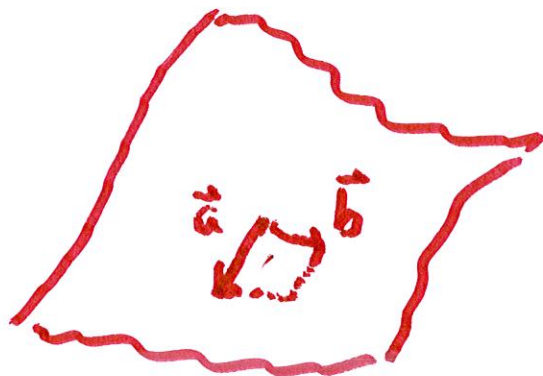
$$= \frac{\pi}{16}$$

$$\therefore \bar{y} = \frac{M_x}{m} = \frac{\pi}{16} \cdot 3 = \frac{3\pi}{16}$$

15.6 Surface Area



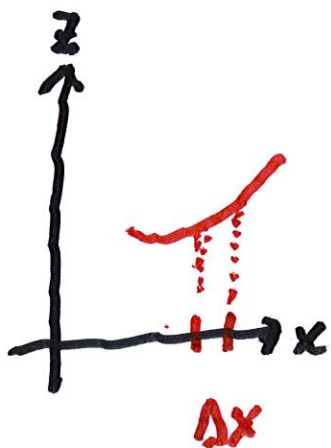
The surface $z = f(x, y)$



$$\vec{a} = \Delta x \vec{i} + f_x \Delta x \vec{k}$$

$$\vec{b} = \Delta y \vec{j} + f_y \Delta y \vec{k}$$

The vector gen. by $\vec{a} \Delta x \times$



$$(\Delta x, f'(x)\Delta x) = \vec{a}$$

Compute area of parallelogram

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= -f_x \Delta x \Delta y \vec{i} - f_y \Delta x \Delta y \vec{j} + \Delta x \Delta y \vec{k}$$

$$|Area| = \left((f_x)^2 + (f_y)^2 + 1 \right)^{\frac{1}{2}} \Delta x \Delta y$$

$$z = x^2 + y^2 = f(x, y)$$

$$f_x = 2x \quad f_y = 2y$$

$$\iint_D \sqrt{1 + 4x^2 + 4y^2} \, dA$$

$$\rightarrow \sqrt{1 + 4x^2 + 4y^2}$$

$$SA = \iint_D \sqrt{1 + 4(x^2 + y^2)} \, dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$

$$= \frac{2\pi}{8} \int_0^2 \sqrt{1+4r^2} \cdot 8r \, dr$$

$$u = 1 + 4r^2$$

$$du = 8r \, dr$$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{u} \, du$$

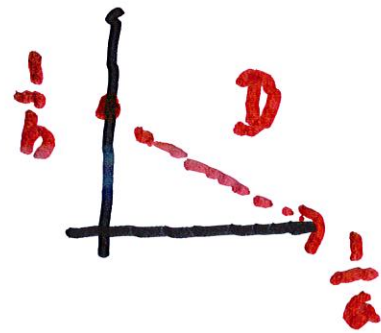
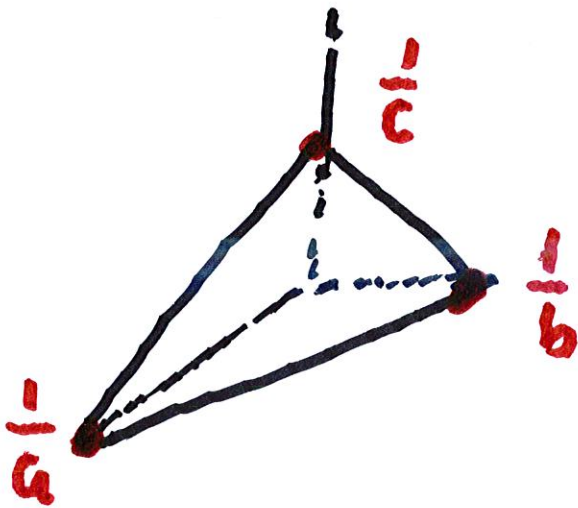
$$= \frac{\pi}{4} \cdot \frac{2}{3} (u)^{3/2} \Big|_1^{17}$$

$$= \frac{\pi}{6} \{ 17^{3/2} - 1 \}$$

Ex. Find surface area of

$$ax + by + cz = 1 \text{ above}$$

the xy -plane in 1st octant



$$z = \frac{1 - ax - by}{c}$$

$$\frac{\partial f}{\partial x} = -\frac{a}{c} \quad \frac{\partial f}{\partial y} = -\frac{b}{c}$$

$$A = \iint_D \sqrt{1 + \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2} dx dy$$

$$= \frac{\sqrt{c^2 + a^2 + b^2}}{c} \iint_D 1 \cdot dA$$



$$= \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{c} \cdot \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2 + b^2 + c^2}}{2abc}$$

