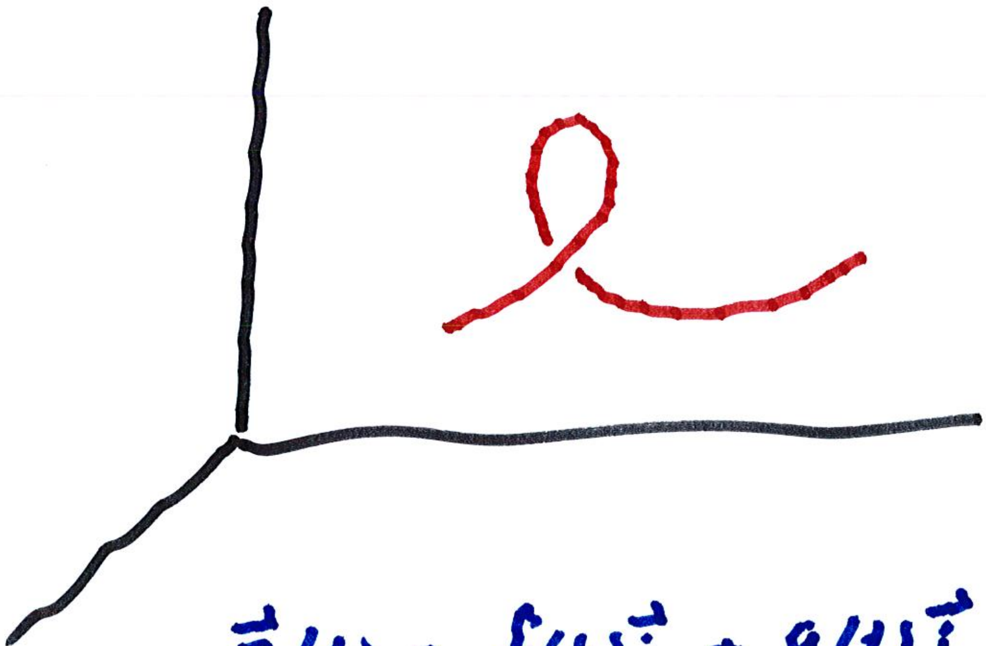
13.3 Arclength and Curvature



$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

$a = t_0 < t_1 \dots < t_j < t_{j+1} \dots < t_N = b$

where $\Delta t = \dfrac{b-a}{N}$

During a Δt-time interval,

$$r(t_{j+1}) - r(t_j) \approx \vec{r}\,'(t_j)\Delta t$$

so the distance traveled is

$$\left| \vec{r}(t_{j+1}) - \vec{r}(t_j) \right| \approx |\vec{r}\,'(t_j)|\Delta t.$$

Adding all distances:

$$\sum_{j=0}^{N-1} \left| \vec{r}(t_{j+1}) - \vec{r}(t_j) \right| = \sum_{j=0}^{N-1} |\vec{r}\,'(t_j)|\Delta t$$

As $N \to \infty$, the total distance is

$$L = \text{Length} = \int_a^b |\vec{r}\,'(t)|\,dt$$

If $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$,

then $\vec{r}\,'(t) = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$

Ex. Compute the length L of the path $\langle \cos t, \sin t, \ln(\cos t) \rangle$

for $0 \le t \le \frac{\pi}{4}$

$$\vec{r}'(t) = \left\langle -\sin t,\ \cos t,\ \frac{-\sin t}{\cos t} \right\rangle$$

$$\therefore L = \int_0^{\pi/4} \sqrt{\sin^2 t + \cos^2 t + \frac{\sin^2 t}{\cos^2 t}}$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 t}\ dt$$

$$= \int_0^{\frac{\pi}{4}} \sec t\, dt = \ln|\sec t + \tan t|\Big|_0^{\frac{\pi}{4}}$$

$$= \ln \left| \sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right) \right|$$

$$- \ln \left| \sec 0 + \tan 0 \right|$$

$$= \ln(\sqrt{2} + 1)$$

Ex. Find length of the path

$$\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} + \frac{t^3}{3}\,\vec{k}$$

for $0 \le t \le 1$.

$$\vec{r}\,'(t) = 2\vec{i} + 2t\vec{j} + t^2\vec{k}$$

$$L = \int_0^1 \sqrt{4 + 4t^2 + t^4}\; dt$$

$$= \int_0^1 \sqrt{(2 + t^2)^2}\; dt$$

$$= \int_0^1 2 + t^2\; dt = 2t + \frac{t^3}{3}\Big|_0^1$$

$$= 2 + \frac{1}{3} = \frac{7}{3}$$

Ex. Express the length of the path $\vec{r}(t) = \langle t^2, t^3, t^4 \rangle$ for $0 \le t \le 2$

$$\vec{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle$$

$$\therefore L = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\, dt$$

This cannot be expressed in terms of standard functions

Ex. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and $3z = xy$ from $(0,0,0)$ to $(6,18,36)$

First let $x = t$. Then $y = \frac{x^2}{2}$

so $y = \frac{t^2}{2}$. Solving for z,

we get $z = \frac{xy}{3}$ or $z = \frac{t \cdot t^2}{3 \cdot 2}$

or $z = \frac{t^3}{6}$.

$x = t$ goes from 0 to 6.

$$\therefore L = \int_0^6 \sqrt{1^2 + t^2 + \frac{t^4}{4}}\, dt$$

$$= \int_0^6 \sqrt{\frac{4 + 4t^2 + t^4}{4}}\, dt$$

$$= \int_0^6 \frac{1}{2} \sqrt{(2 + t^2)^2}$$

$$= \int_0^6 \frac{1}{2} (2 + t^2)\, dt$$

$$= \int_0^6 1 + \frac{t^2}{2} \, dt$$

$$= t + \frac{t^3}{6} \Big|_0^6 = 6 + 36 = \underline{\underline{42}}$$

Ex. Reparametrize the path

$$t \longrightarrow \vec{r}(t) = \left\langle 2t - 3,\ 4t + 2,\ -4t + 5 \right\rangle$$

First we compute the

distance function ~~of the~~

~~distance~~ $s(t)$ of the path

for $\vec{r}\,'(t) = \langle 2, 4, -4 \rangle$

$$\therefore \quad s(t) = \int_0^t \sqrt{36}\, du = 6t$$

Hence $s = 6t \rightarrow t = \frac{s}{6}$

$\therefore$ We set $\vec{r}(s)$ by

$$\vec{r}(s) = \left\langle 2\left(\frac{s}{6}\right) - 3,\ 4\left(\frac{s}{6}\right) + 2,\ -4\left(\frac{s}{6}\right) + 5 \right\rangle$$

$$\vec{r}(s) = \left\langle \frac{s}{6} - 3,\ \frac{2s}{3} + 2,\ -\frac{2s}{3} + 5 \right\rangle$$

Ex. Compute the curvature of

$$\vec{r}(t) = 2\cos t\,\vec{i} - 2\sin t\,\vec{j} + 4t\,\vec{k}$$

There are 2 formulas

$$K(t) = \frac{|T'(t)|}{|r'(t)|}$$

or $\kappa(t) = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

We use the second.

$\vec{r}\,'(t) = -2\sin t\,\vec{i} - 2\cos t\,\vec{j} + 4\vec{k}$

$\vec{r}\,''(t) = -2\cos t\,\vec{i} + 2\sin t\,\vec{j}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & -2\cos t & 4 \\ -2\cos t & 2\sin t & 0 \end{vmatrix}$$

$$= -8\vec{i} - 8\vec{j} - (-4\sin^2 t - 4\cos^2 t)$$

$$= -8\vec{i} - 8\vec{j} + 4\vec{k}$$

Note

$$|-8\vec{i} - 8\vec{j} + 4\vec{k}|$$

$$= \sqrt{64 + 64 + 16} = 12$$

and $|\vec{r}'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 16}$

$$= \sqrt{20}$$

$$\therefore K(t) = \frac{12}{(\sqrt{20})^3} = \frac{12}{20\sqrt{20}}$$

$$= \frac{3}{10\sqrt{5}}$$

$$\vec{r}(t) = \langle 2t-3, -4t+5, -4t+1 \rangle$$

$$= t\langle 2, 4, -4 \rangle + \langle -3, 5, 1 \rangle$$

$$\vec{r}\,' = \langle 2, 4, -4 \rangle$$

$$\vec{r}\,'' = \vec{0} \quad \cancel{\vec{r}\,''(t) = 0}$$