13.4 Velocity and Acceleration

Suppose the position of a particle at time t is $\vec{r}(t)$.

After h seconds, the distance it travels is $|\vec{r}(t+h) - \vec{r}(t)|$

$\therefore$ The speed is roughly

$$\frac{|\vec{r}(t+h) - \vec{r}(t)|}{h} = \left| \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \right|.$$

As $h \to 0$, the limit is the speed

$\left| \vec{r}\,'(t) \right|$. If we focus on

$\vec{r}\,'(t)$, we get a vector

that gives the direction of

motion and its speed.

Thus, $\vec{r}\,'(t)$ is the <u>velocity</u> of

the particle.

Similarly, $\vec{v}'(t) = \vec{a}(t)$ is

the acceleration of the particle.

We can rewrite this as

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t).$$

Also $|\vec{v}(t)| = |\vec{r}'(t)|$ is

the speed.

Ex. Suppose $\vec{r}(t) = (2t+1, 3t^2)$

is the position at time t.

Compute it's velocit, speed, and

acceleration at time t:

$\vec{r}\,'(t) = \langle 2, 6t \rangle$ = velocity $\vec{v}(t)$

$|\vec{v}(t)| = \sqrt{4 + 36t^2}$ = speed

$\vec{a}(t) = \vec{r}\,''(t) = \langle 0, 6 \rangle$ = acceleration

Ex. If $\vec{r}(t) = (t^2+1)\vec{i} - t^3\vec{j} + 2t^2\vec{k}$

is the position of a particle

compute the velocity vector, the

speed, and the acceleration

$$\vec{v}(t) = \vec{r}'(t) = 2t\,\vec{i} - 3t^2\,\vec{j} + 4t\,\vec{k}$$

$$\therefore \text{Speed} = \sqrt{4t^2 + 9t^4 + 16t^2}$$

$$= \sqrt{20t^2 + 9t^4}$$

$$\vec{a}(t) = \vec{r}''(t) = 2\cancel{t}\,\vec{i} - 6t\,\vec{j} + 4\vec{k}$$

is the acceleration

We can go in the other direction

and assume we know the

acceleration:

Ex. Suppose the acceleration

of a particle is

$$\vec{a}(t) = (2t-1)\vec{i} + 3t^2\vec{j} + (2-t)\vec{k}$$

and also that $\vec{v}(0) = \vec{i} + 2\vec{j}$

and $\vec{r}(0) = j - \vec{k}$

Find the position $\vec{r}(t)$

$$\vec{v}'(t) = \vec{a}(t), \quad \text{so } \vec{v}(t) = \int \vec{a}(t)\,dt$$

Hence $\vec{v}(t) = (t^2 - t)\vec{i} + t^3\vec{j} + \left(2t - \frac{t^2}{2}\right)\vec{k} + \vec{C}$

$$\vec{v}(0) = \vec{0} + \vec{C}$$

$$\therefore \vec{C} = \vec{i} + 2\vec{j}$$

$$\rightarrow \vec{v}(t) = (t^2 - t + 1)\vec{i} + \left(\overset{t^3+2}{\cancel{3t^3+2}}\right)\vec{j} + \left(2t - \frac{t^2}{2}\right)\vec{k}$$

Similarly,

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\overset{\frac{t^4}{4}+2t}{\cancel{t^3+2t}}\right)\vec{j} + \left(t^2 - \frac{t^3}{6}\right)\vec{k} + \vec{C}$$

$\vec{C} = \qquad = \vec{j} - \vec{k}$, so

$$\vec{r}(t) = \left(\frac{t^3}{3} - \frac{t^2}{2} + t\right)\vec{i} + \left(\overset{\frac{t^4}{4}}{\cancel{t^3}} + 2t + 1\right)\vec{j} + \left(t^2 - \frac{t^3}{6} - 1\right)\vec{k}$$

In general, many problems in physics start with the force $\vec{F}$ known. Since $\vec{F} = m\vec{a}$, this leads to knowing $\vec{a}$.

Ex Suppose $\vec{r}(t) = a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}$

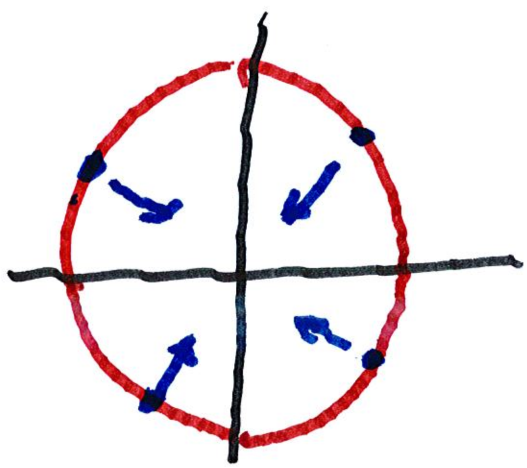
Find $\vec{a}(t)$.

$$\vec{r}\,'(t) = -a\omega\sin\omega t\,\vec{i} + a\omega\cos\omega t\,\vec{j}$$

Hence .

$$\vec{a}(t) = -a\omega^2 \cos\omega t\,\vec{i} - a\omega^2 \sin\omega t\,\vec{j}$$

$$= -\omega^2 \left(a\cos\omega t\,\vec{i} + a\sin\omega t\,\vec{j}\right)$$



we see that

$$\vec{a}(t) = -\omega^2 \vec{r}(t)$$

This indicates that force acting on the particle points inward.

Motion on earth.

Suppose a projectile is fired

at $t=0$, with $\vec{r}(0) = \vec{0}$

It's initial velocity is

(1) $\vec{v}(0) = V_0 (\cos\alpha \, \vec{i} + \sin\alpha \, \vec{j})$

The force acting on the particle

is $\vec{F} = m\vec{a} = -mg\vec{j}$

$\therefore$ acceleration $= \vec{a} = -g\vec{j}$

$\rightarrow \vec{v}(t) = -gt\vec{j} + \vec{C} = -gt\vec{j} + \vec{v}(0)$

$\rightarrow \vec{r}(t) = \dfrac{-gt^2}{2}\vec{j} + t\vec{v}(0) + \vec{D}.$

Since $\vec{r}(0) = \vec{0}$, $\vec{D} = 0$.

Recall (1), we get

$\vec{r}(t) = v_0 t\cos\alpha\,\vec{i} + \left(v_0 t\sin\alpha - \dfrac{gt^2}{2}\right)\vec{j}$

Review Problem

Suppose $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

$$K = \text{Curvature} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$\vec{r}\,' = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= 6t^2\,\vec{i} - 6t\,\vec{j} + 2\,\vec{k}$$

$$\left| \quad {}'' \qquad\qquad {}'' \quad \right| = \sqrt{36t^4 + 36t^2 + 4}$$

$$\left| \vec{r}\,'(t) \right| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\therefore K = \frac{\sqrt{36t^4 + 36t^2 + 9}}{\left(1 + 4t^2 + 9t^4\right)^{3/2}}$$

$$K = \frac{1}{a}$$

$$K = \frac{T'(t)}{\left|\cancel{T'(t)}\right| \left|r'(t)\right|} \quad \left|T'(t)\right|$$

$$K = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

Ex. Find length of $\vec{r}(t) = \langle 1, t^2, t^3 \rangle$

for $0 \le t \le 1$

$$\vec{r}\,'(t) = \langle 0, 2t, 3t^2 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4t^2 + \cancel{36}\,9t^4}$$

$$L = \int_0^1 \sqrt{4t^2 + 36t^4}\, dt$$

$$= \int_0^1 \sqrt{4 + 36t^2}\; t\, dt$$

$u = 4 + 36t^2 \qquad u(0) = 4 \qquad u(1) = 40$

$du = 72t\,dt$

$\therefore t\,dt = \frac{1}{72}\,du$

$$\int = \int_4^{40} \sqrt{u} \cdot \frac{du}{72}$$

$$= \frac{2}{3 \cdot 72} u^{3/2} \Big|_4^{40}$$

$$= \frac{1}{108} \left(40^{3/2} - 8 \right)$$

Ex. Reparameterize wrt arclength

$$\vec{r}(t) = 2t\,\vec{i} + (1-3t)\,\vec{j} + (5+4t)\,\vec{k}$$

$$\vec{r}\,'(t) = \langle 2, -3, 4 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{4+9+16} = \sqrt{29}$$

$$\therefore\ s(t) = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$$

$$\Rightarrow s = \sqrt{29}\, t$$

$$\text{or} = t = \frac{s}{\sqrt{29}}$$

$$\vec{r}(t(s)) = \left\langle \frac{2s}{\sqrt{29}},\ \left(1 - \frac{3s}{\sqrt{29}}\right),\ 5 + \frac{4s}{\sqrt{29}} \right\rangle$$

$$= s\left\langle \frac{2}{\sqrt{29}},\ \frac{-3}{\sqrt{29}},\ \frac{4}{\sqrt{29}} \right\rangle + \text{—}$$