14.1 Functions of Several Variables

We often think of a surface

as $z = x^2 y$ or $z = y + x^2$

and we write

$f(x,y) = x^2 y$ or $g(x,y) = y + x^2$.

Instead of variable x

$f(x,y)$ depends on 2 variables

Def'n A function f of two variables is a rule that assigns to each ordered pair of numbers (x,y) in a set D a unique real number denoted by $f(x,y)$. D is the domain of f and its range is the set of values that f takes on, i.e.,

$$R_f = \{ f(x,y) \mid (x,y) \in D \}$$

If the domain D is not specified, then the domain is the set of (x,y) such that $f(x,y)$ is well-defined.

Ex. Find the domain and range of $f(x,y) = \dfrac{\ln(x^2+y-2)}{x-2}$

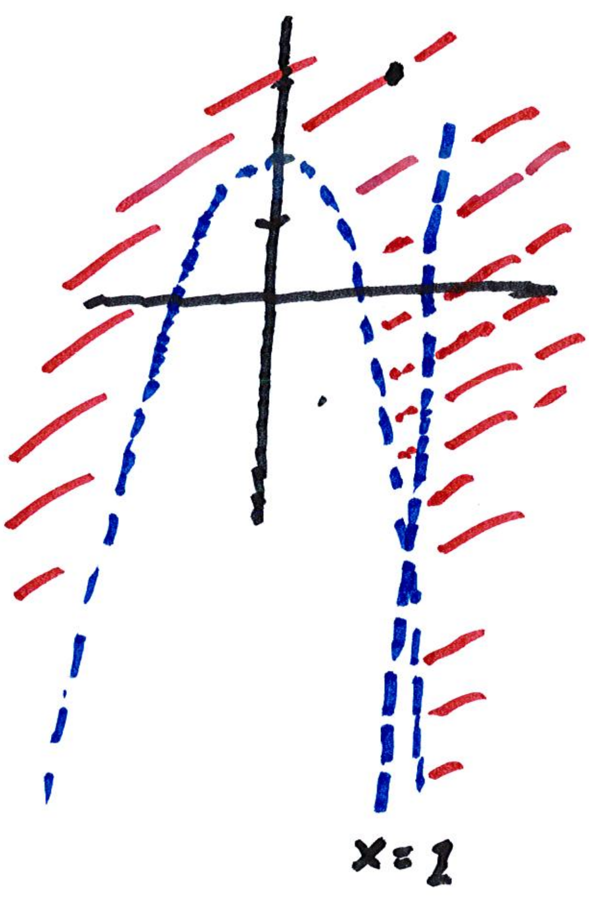
For $\ln(x^2+y-2)$, we need

$x^2 + y - 2 > 0$, i.e.,

$y > 2 - x^2$.

For the denominator $x-2$,

we need $x-2 \neq 0$ or $x \neq 2$.



$D: \{(x,y) \mid y > 2 - x^2$

and $x \neq 2 \}$

Look at $x=0$

and $y > 2$

$$f(0, y) = \frac{\ln(y-2)}{-2}$$ takes on all values.

$\therefore R_f = \{\text{all real numbers}\}$

Ex. Let $g(x, y, z)$

$$= x^3 y^2 z \sqrt{10 - x - y - z}$$

We need $10 - x - y - z \geq 0$,

i.e., $\underbrace{x + y + z \leq 10}_{\text{Domain } D}$

Ex. $P(x,y) = 100\,x^{\frac{1}{4}}y^{\frac{3}{4}}$.

We need $x \geq 0$ and $y \geq 0$

$\therefore\ D = \{(x,y);\ x \geq 0 \text{ and } y \geq 0\}$

Note that

$P(x,1) = 100x^{1/4} \geq 0$ for all $x \geq 0$

$\therefore\ R_p = \{x;\ x \geq 0\}$

Ex. If f is a function of two variables with domain D,

then the graph of f is the set of all (x, y, z) such that

$z = f(x, y)$ and $(x, y) \in D$.

Ex. Find the graph of

$f(x, y) = 2\sqrt{x^2 + y^2}$

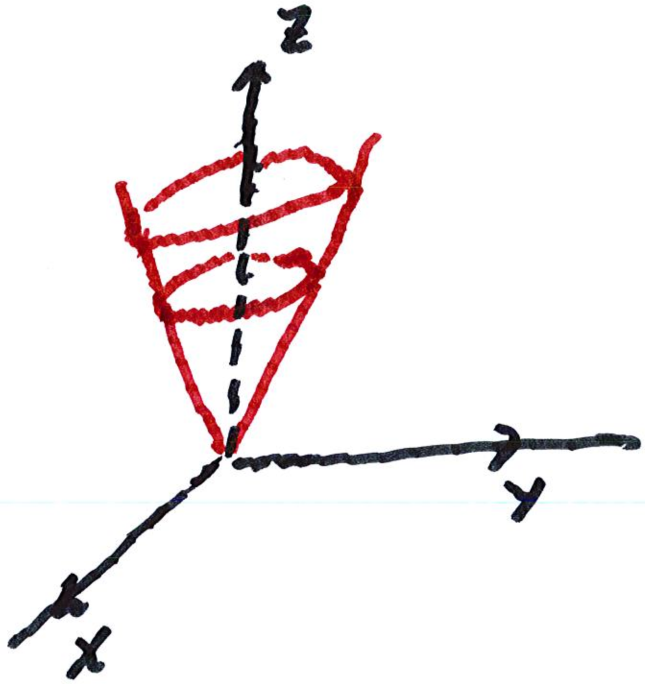
any (x, y)

$z = 2\sqrt{x^2 + y^2}$

or

$z^2 = 4(x^2 + y^2)$

$z \geq 0$



Note that

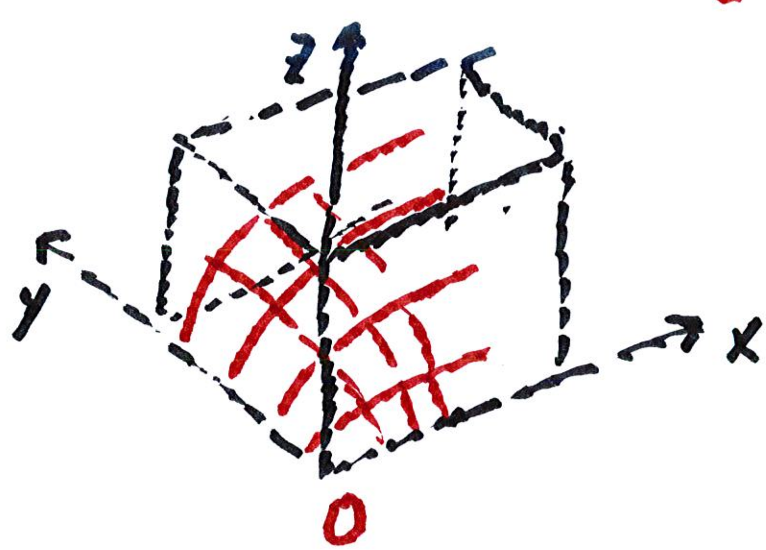
z must be

$z \geq 0$

Sketch the graph of

$f(x, y) = x^{1/4} y^{3/4}$

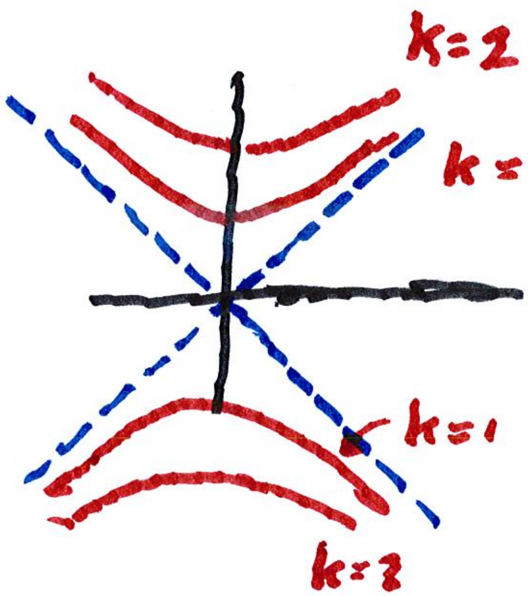
$z = x^{1/4} y^{3/4}$

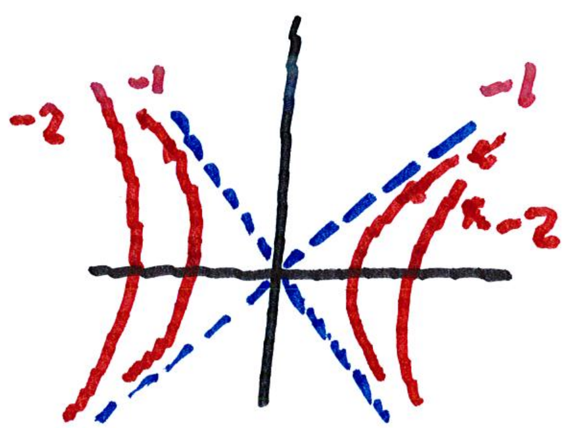


Level Curves of $f(x,y)$

A level curve is $= \{(x,y) \mid f(x,y) = k\}$

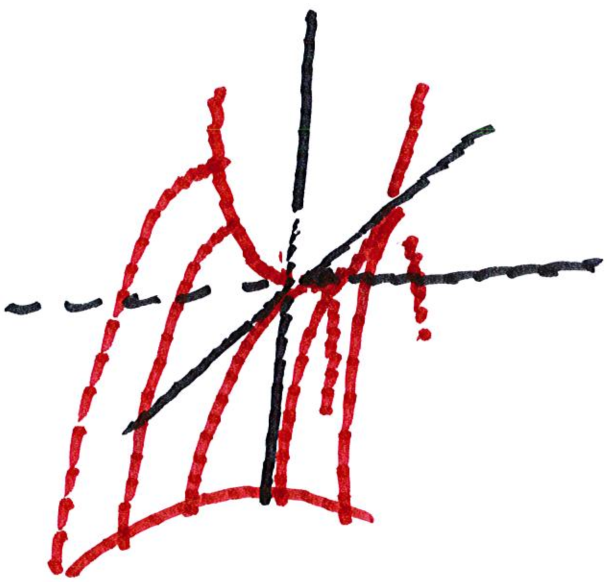
Ex. If $f(x,y) = \{(x,y) \mid y^2 - x^2 = k\}$





Find Graph of $f(x,y) = y^2 - x^2$

$\rightarrow \quad z = y^2 - x^2$

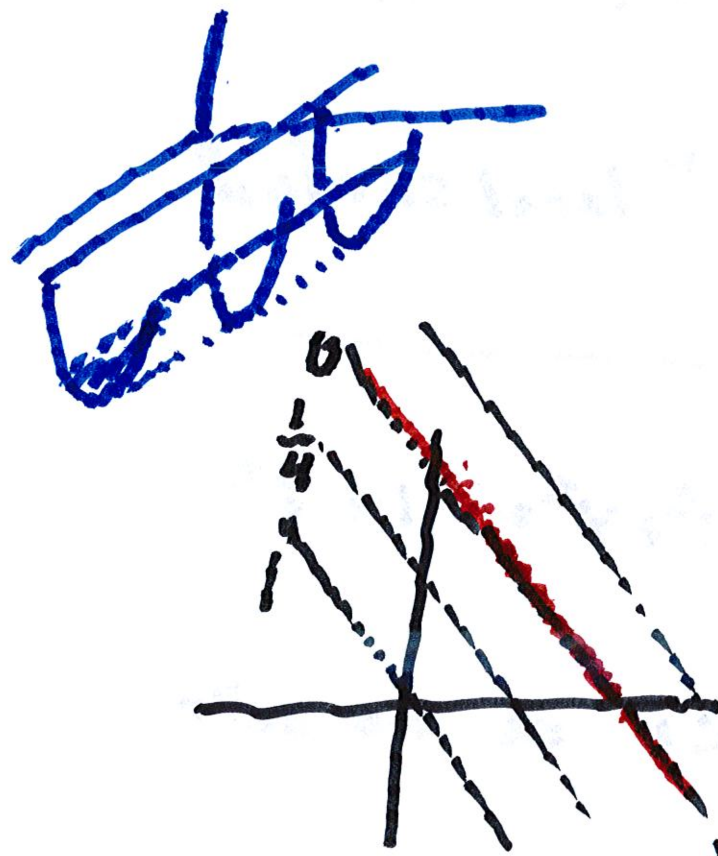


Saddle point at origin.

Find the level curves and the graph of $f(x,y) = (1-x-y)^2$

Sketch the graph of

$f(x,y) = (x+y-1)^2$



$x+y-1 = 0$

Sketch the level surface of

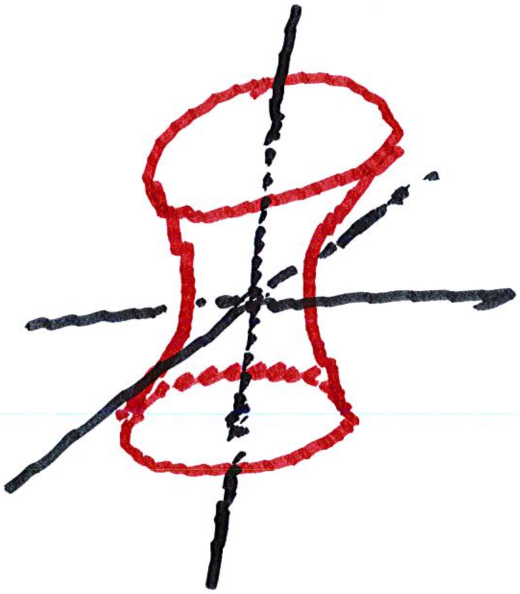
$$x^2 + y^2 - z^2 = k$$

If $k = 0 \rightarrow z^2 = x^2 + y^2$

A cone is rotated about the z-axis

If $k > 0$, say $k = 1$

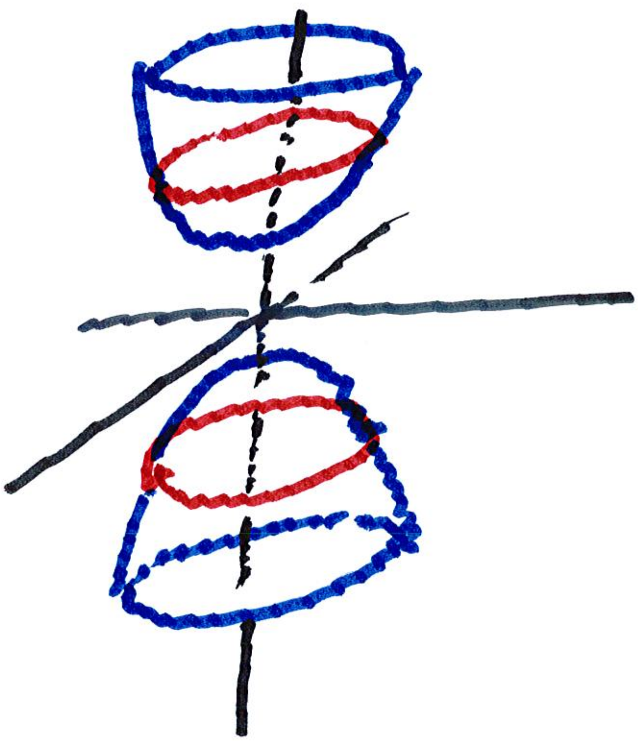
$$x^2 + y^2 = 1 + z^2$$



hyperboloid of 1 sheet.

Now suppose that $k < 0$,

say $k = -1$



hyperboloid of 2 sheets

Ex. Find the level surfaces

$$\left(\text{The set of } (x, y, z) \mid x^2 + y^2 + z^2 = k \right)$$

If $k < 0$, no solution at all

If $k = 0$, $x^2 + y^2 + z^2 = 0$

(the origin)

If $k > 0$, $x^2 + y^2 + z^2 = k$

$\rightarrow$ sphere of radius $\sqrt{k}$

$x = x(t)$, $y = y(t)$.

Then the acceleration is

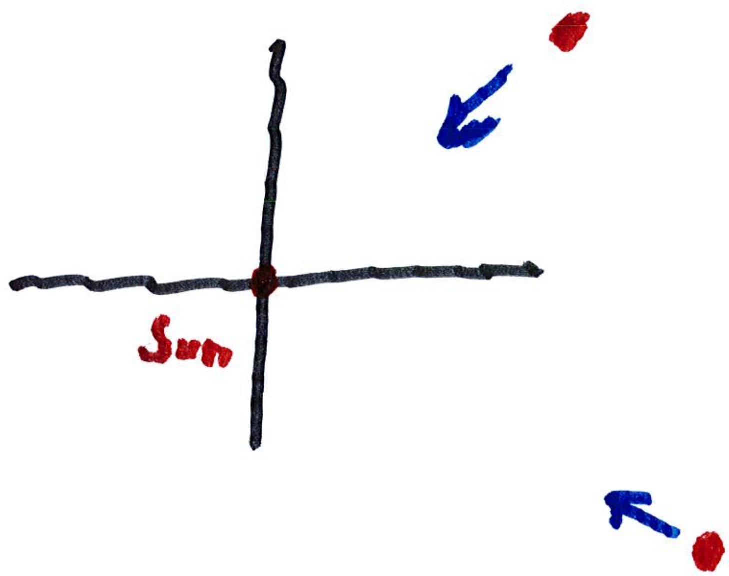
$x''(t)$, $y''(t)$

$m\vec{a}(t) = mx''$, my'' (Set $m = 1$)

Gravitational Force is

$$-\frac{x''}{\sqrt{x^2+y^2}}$$

$$F_g = \left(\frac{-x}{(x^2+y^2)^{3/2}} , \frac{-y}{(x^2+y^2)^{3/2}} \right)$$



F_g points to sun, magnitude

is $\propto \frac{1}{r^2}$

$$x''(t) = \frac{-x}{(x^2+y^2)^{3/2}}$$

$$y''(t) = \frac{-y}{(x^2+y^2)^{3/2}}$$

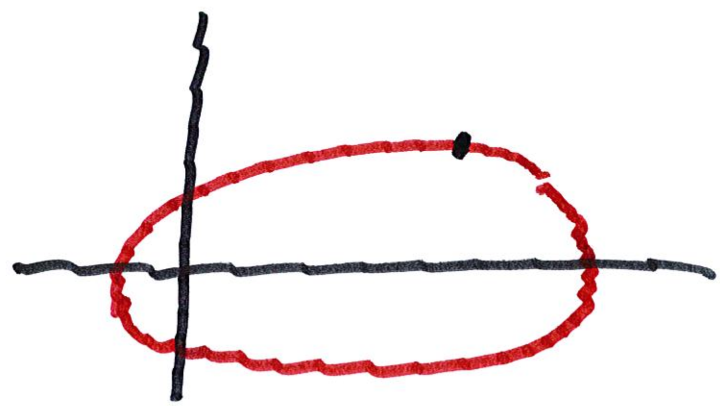
A second order system

Newton showed $(x(t), y(t))$

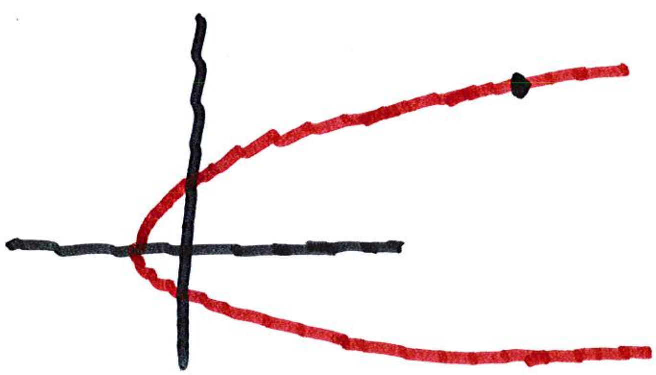
travels along either an

ellipse, a parabola,

or a hyperboloid

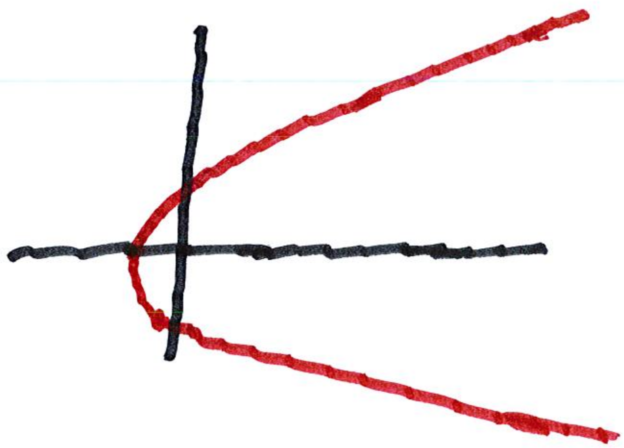


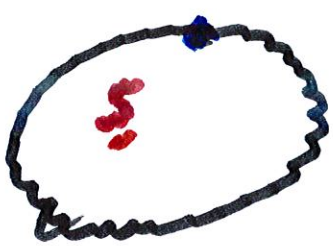
ellipse

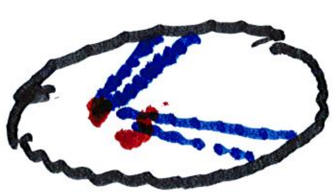


parabola

or a hyperbola:







T: Time of 1 orbit

$$T = k a^{3/2}$$