15.8 Triple Integrals in Cylindrical Coordinates

In the plane we can use polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In 3 dimensions, we can use a related system of polar coordinates

If (r, θ, z) are cylindrical coord.

then the rectangula coord. are

$x = r\cos\theta \quad y = r\sin\theta \ , \ z = z$

and if (x, y, z) are rectangular

coord., then the cylindrical coord are

$r^2 = x^2 + y^2 \ , \ \tan\theta = y/x \ , \ z = z.$

Ex. If a point P has cylindrical coord. $= \left(2, \frac{\pi}{6}, 3\right)$,

then the rectangular coord. are

$$x = 2 \cdot \cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2 \sin\frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$$

$$z = 3$$

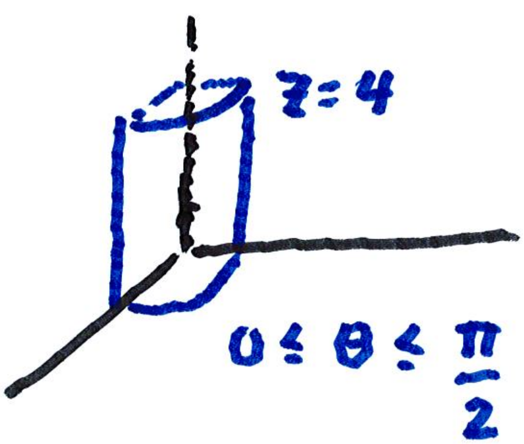
Ex. What are the cylindrical coord. of the surface

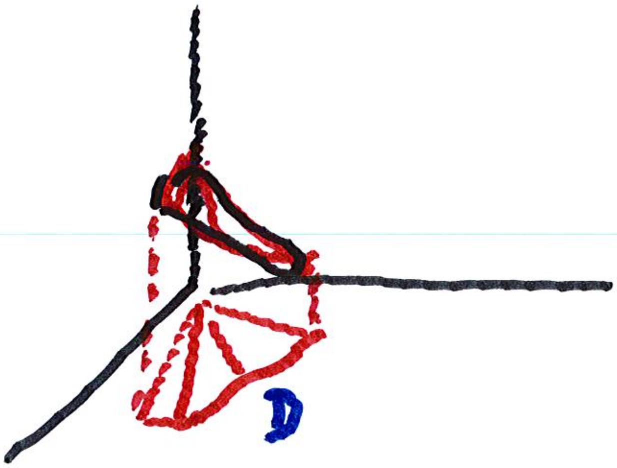
Ex: $x^2+y^2=6$, in the first octant

with $0 \le z \le 4$

$r^2=6$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le z \le 4$

$\therefore (r, \theta, z) = (\sqrt{6}, 0 \le \theta \le \frac{\pi}{2}, 0 \le z \le 4$





$$\iiint_E f(x,y,z)\,dV$$

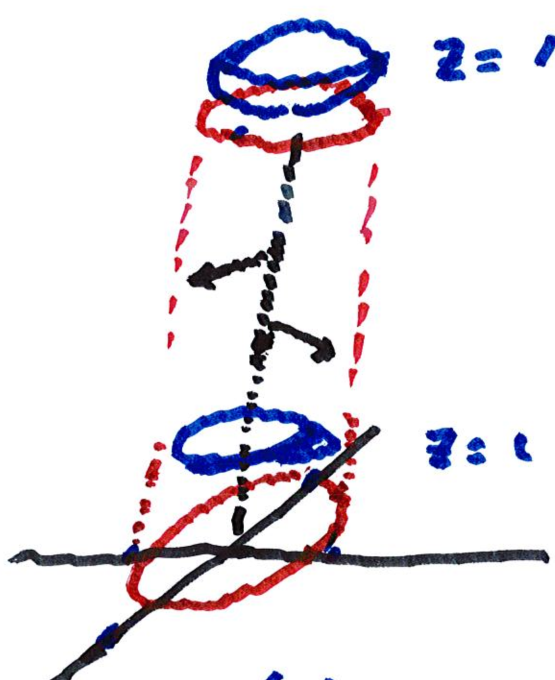
$$= \iint_D \left[\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(r\cos\theta, r\sin\theta, z)\,dz \right] dA$$

$dA = r\,dr\,d\theta$

Ex. Find $\iiint\limits_{E} z \, dA$,

where E is bounded by

$$1 \le z \le x^2+y^2+2 \quad , \quad x^2+y^2=1$$



$$\iiint z \, dV = \int_0^{2\pi} \int_0^1 \int_1^{r^2+2} z \, dz \, r \, dr \, d\theta$$

$dz \; r \, dr \; d\theta$

$$= 2\pi \int_0^1 \int_1^{r^2+2} z \, dz \, r \, dr$$

$$= 2\pi \int_0^1 \left. \frac{z^2}{2} \right|_1^{r^2+2} r \, dr$$

$$= \pi \int_0^1 \left((r^2+2)^2 - 1 \right) r \, dr$$

$$= \pi \int_0^1 r^5 + 4r^3 + 3r \quad dr$$

$$= \pi \left(\frac{r^6}{6} + r^4 + \frac{3r^2}{2} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{6} + 1 + \frac{3}{2} \right) = \frac{5\pi}{3}$$

Ex $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)^{\frac{1}{2}}\,dV$

$y=\sqrt{4-x^2}$

$y^2=4-x^2$

$$=\int_0^{\pi}\int_0^{2}\int_r^{2} r \;\; r\,dr\,d\theta$$

dz

$$=\pi\int_0^{2}\int_r^{2} r^2\,dr$$

r $\quad$ z

$$\pi \int_0^2 r^2 z \Big|_r^2 \, dr$$

$$= \pi \int_0^2 (r^2 \cdot 2 - r^2 \cdot \overset{r^2}{0}) \, dr$$

$$= \pi \int_0^2 2r^2 \, dr = \frac{\pi \cdot 2r^3}{3} \Big|_0^1 = \frac{16\pi}{3}$$

$r^4 = 2 - r^2$

$r^2 = 2 \quad \therefore r = \sqrt{2}.$

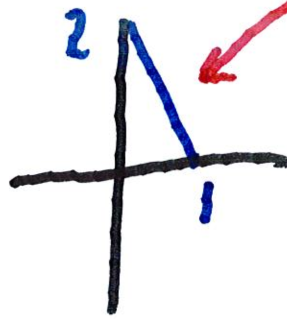
Ex. Suppose E = solid tetrahedron bounded by $2x + y + 2z = 2$, $x = 0$, $y = 0$ and $z = 0$.

If the density of E $= x$, calculate mass of E.

Set $z = 0$

$\rightarrow 2x + y = 2$

$\rightarrow y = 2 - 2x$



$$\text{Mass} = \int_0^1 \int_0^{2-2x} \int_0^{1-x-\frac{y}{2}} x \, dz\, dy\, dx$$

$$= \int_0^1 x \int_0^{2-2x} \left(1 - x - \frac{y}{2}\right) dy\, dx$$

Note that the tetrahedron sits

on D and D sits on $[0, 2]$

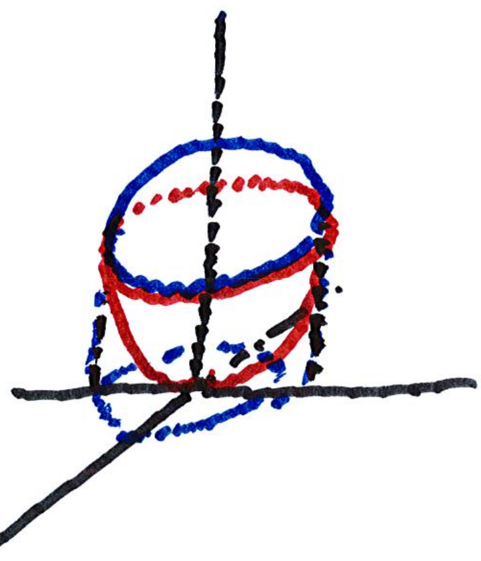
$$\therefore \iiint_E 2z \, dV$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{1-\frac{x}{2}-y} 2z \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} z^2 \Big|_{z=0}^{z=1-\frac{x}{2}-y} dy \, dx$$

Ex. Find the integral $\iiint\limits_E zx \, dV$

where E is the region in the first octant bounded by $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$



Solid lies above a disc in the xy-plane

It's radius is

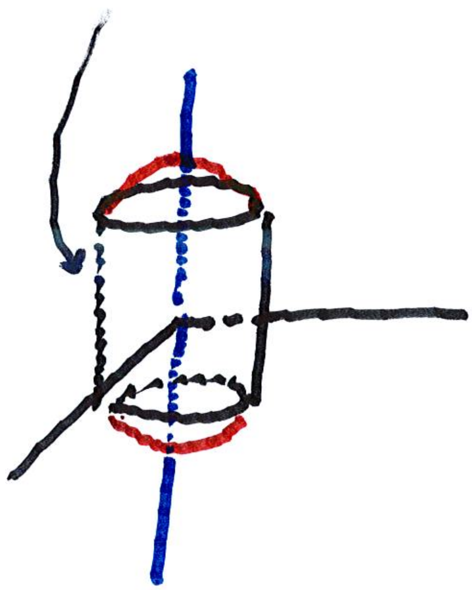
$$\sqrt{2-r^2} = z = r^2$$

Ex. Find the volume of the solid

that lies within both the cylinder

$x^2+y^2=1$ and the sphere

$x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$



region satisfies

$0 < r < 1$ $\quad 0 \le \theta \le 2\pi$

and $-\sqrt{4-x^2-y^2} < z$

$< \sqrt{4-x^2-y^2}$