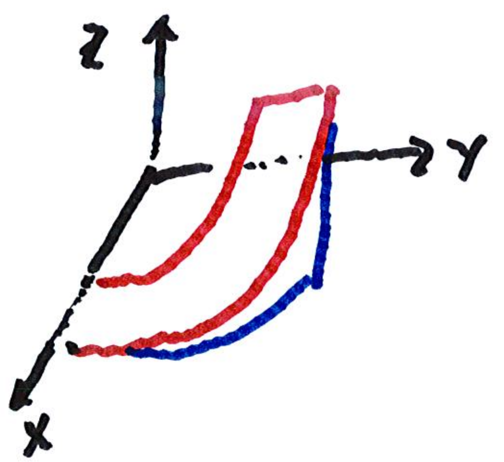
Ex. Compute $\displaystyle\iiint_E z\,dV$ where E

is in the first octant and lies

between $x^2+y^2=1$ and $x^2+y^2=4$

and lies below $z=y$



$$\iiint_E z\,dV$$

$$= \int_0^{\frac{\pi}{2}} \int_1^{2} \int_0^{r\sin\theta} z \; dz\; r\,dr\; d\theta$$

θ $\quad r \quad$ z

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{z^2}{2} \Big|_0^{r\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \frac{r^3 \sin^2\theta}{2}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{r^4}{8} \sin^2\theta \Big|_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(4 - \frac{1}{4}\right) \frac{\sin^2\theta}{2}\,d\theta$$

$$= \frac{15}{8} \int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = \frac{15}{8} \int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{15\pi}{16} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta$$

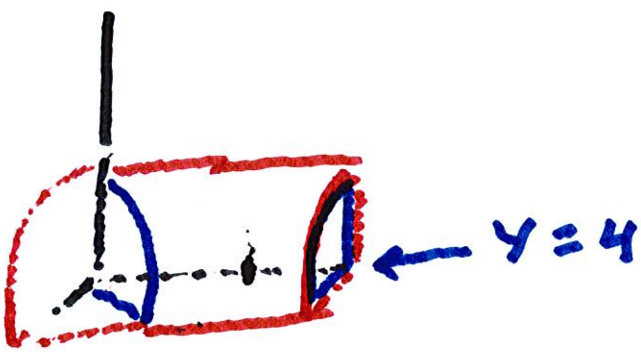
$$= \frac{15}{32}\pi - \left(\frac{15 \sin 2\theta}{32} \Big|_0^{\frac{\pi}{2}} \right)$$

$$= \frac{15\pi}{32}$$

Ex. Let E be the region in the first octant bounded by $x^2 + z^2 = 4$ and bounded between $y = 4$ and $y = x$.

Use cylindrical coordinates to

express $\displaystyle\iiint_E xy \, dV$.



We use cylindrical coord.

with $x^2 + z^2 = r^2$ $\quad x = r\cos\theta$

$z = r\sin\theta$

y is like z. The variable y

satisfies $x \le y \le 4$

or, $r\cos\theta \le y \le 4$

$$\iiint_E xy\,dV = \int_0^{\frac{\pi}{2}} \int_0^2 \int_{r\cos\theta}^4 r^2 \cos\theta \sin\theta$$

$$dV = \underbrace{dy\ r\,dr \cdot d\theta}$$

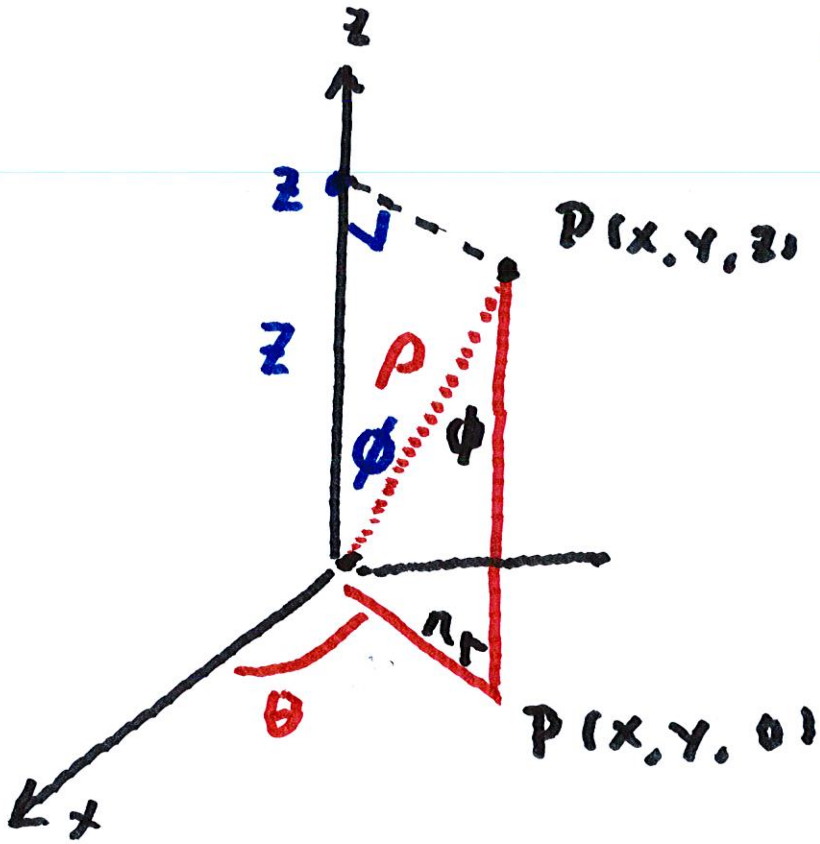
$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_{r\cos\theta}^4 r^3 \cos\theta \sin\theta$$

$dy\,dr\,d\theta$

$\theta \quad r \quad y$

Spherical Coord.

ϕ = angle between ρ and positive z-axis



$$\frac{z}{\rho} = \cos\phi$$

$$z = \rho\cos\phi$$

$x = r\cos\theta \quad y = r\sin\theta$

$$\frac{r}{\rho} = \sin\phi \rightarrow r = \rho\sin\phi$$

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta$

Spherical Coord (ρ, θ, ϕ)

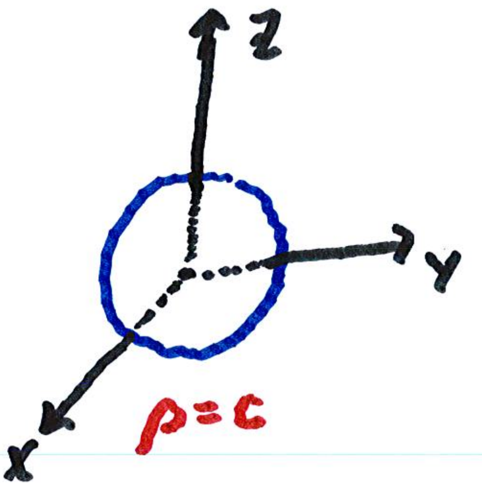
$\rho = \sqrt{x^2+y^2+z^2}$ or $\rho^2 = x^2+y^2+z^2$

If we project (x, y, z) down to

the xy-plane, then

θ = usual coordinate of $(x, y, 0)$

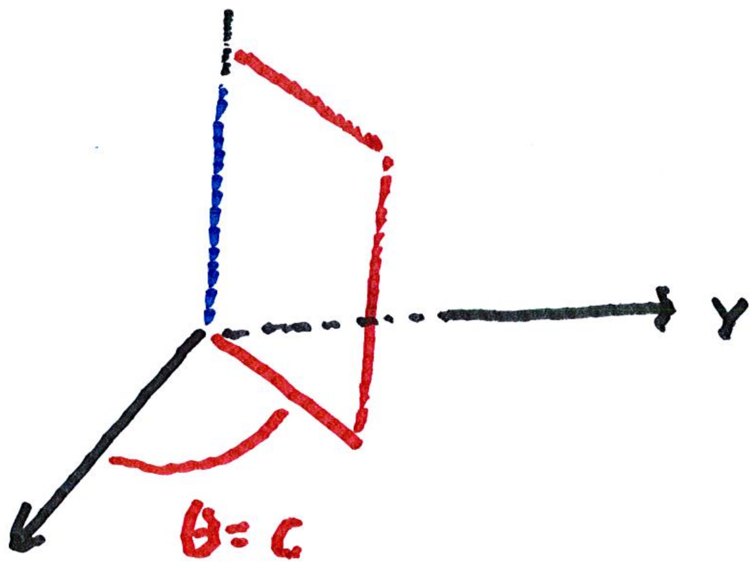
$x = r\cos\theta$, $y = r\sin\theta$



$\rho = c$

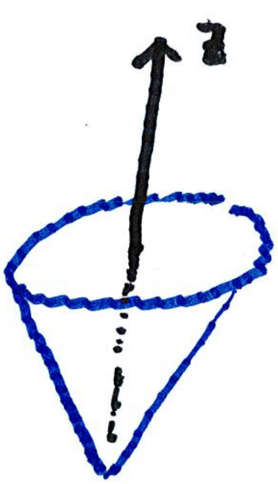
sphere

(ρ, θ, ϕ) are spherical coord:



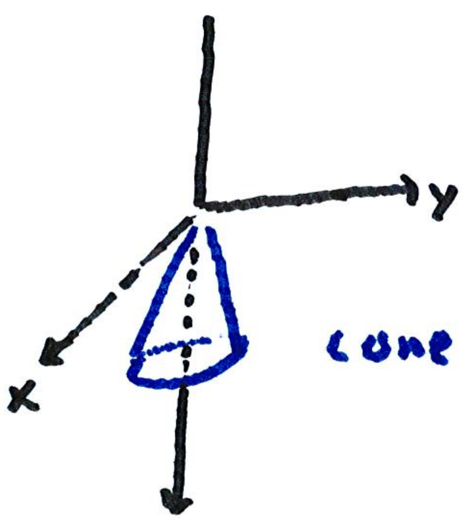
$\theta = c$

half-plane



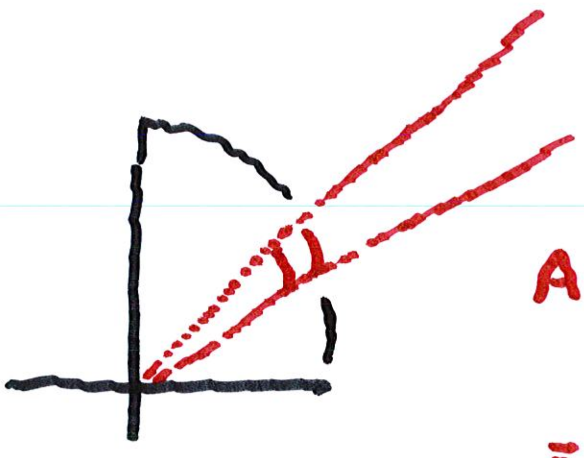
$\phi = c < \frac{\pi}{2}$ cone

$\frac{\pi}{2} < \phi < \pi$



cone

In polar coord



Area of $\Delta_r \Delta_\theta$

$= r \Delta_r \Delta_\theta$

In spherical coord.

Eucl. Volume of $\Delta_\rho \Delta_\theta \Delta_\phi$

$= \rho^2 \sin\phi$

$\therefore$ When converting from

(x, y, z) integral to a spherical,

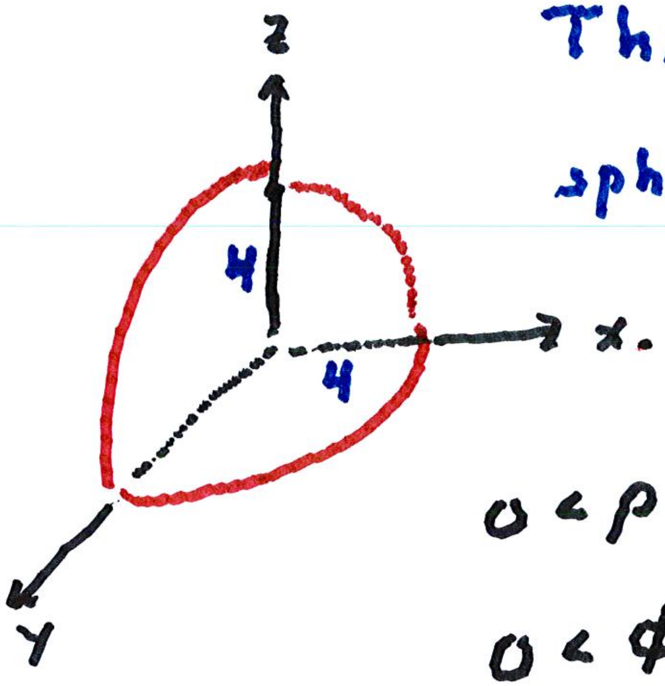
You must include $\rho^2 \sin\phi$.

Ex. Let E = region of ball

(about $(0,0,0)$) of radius 4

in the first octant.

Compute $\displaystyle\iiint_E z\, dV$.



This is a box in spherical coord.

$0 < \rho < 4$

$0 < \phi < \frac{\pi}{2}$

$0 < \theta < \frac{\pi}{2}$

$$\iiint_E z \, dV = \int_0^4 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho\cos\phi \; \rho^2 \sin\phi \; d\phi \, d\rho \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi \frac{\rho^4}{4} \, d\phi \, d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi \, d\phi \, d\theta$$

$$= 64 \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2\phi \Big|_0^{\frac{\pi}{2}} \, d\theta$$

$$= 32 \int_0^{\frac{\pi}{2}} d\theta = \underline{\underline{16\pi}}$$

Find the center of mass of

$E = \{ (x, y, z) \quad 1 \leq x^2 + y^2 + z^2 \leq 4,$

where $z \geq 0 \}$

Assume density = 1

$1 \leq \rho^2 \leq 4 \rightarrow 1 \leq \rho \leq 2$

$z \geq 0 \rightarrow \quad 0 \leq \phi \leq \frac{\pi}{2}$

$$\text{mass} = \underset{\theta}{\int_0^{2\pi}} \underset{\phi}{\int_0^{\frac{\pi}{2}}} \underset{\rho}{\int_1^{2}} 1 \cdot \rho^2 \sin\phi$$

$$\iiint_E = 2\pi \int_0^{\frac{\pi}{2}} \int_1^2 \rho^2 \sin\phi$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\phi \left. \frac{\rho^3}{3} \right|_1^2$$

$$= \frac{2\pi}{3} \cdot (8-1) \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi$$

$$= -\frac{14\pi}{3} \int_0^{\frac{\pi}{2}} -\sin\phi \, d\phi$$

$$= \frac{-14\pi}{3} \cos\phi \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{-14\pi}{3} (0 - 1) = \frac{14\pi}{3}.$$

$$M_{xy} = \iiint_E z \, dV$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^2 \rho\cos\phi \;\; \rho^2 \sin\phi$$

$$d\rho \, d\phi \, d\theta$$

$$= \frac{45}{28} \qquad \bar{y} = 0 \text{ and } \bar{x} = 0$$

by symmetry

$$\therefore (\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{45}{28}\right)$$

Ex. Find the volume of the region below the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{x^2+y^2}$

Set $r=\sqrt{x^2+y^2}$

$z^2 = 1-r^2$

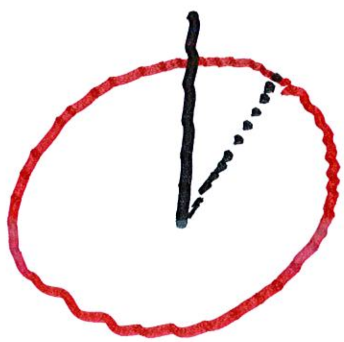
and $z = r \rightarrow z^2 = r^2$

$\therefore \quad 1-r^2 = r^2 \rightarrow 2r^2 = 1 \rightarrow r = \frac{1}{\sqrt{2}}$

$z = r = \frac{1}{\sqrt{2}}$

$$\therefore \tan\phi = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$\Rightarrow \phi = \frac{\pi}{4}$$



$$V = \underset{\theta}{\int_0^{2\pi}} \underset{\phi}{\int_0^{\pi/4}} \int_0^1 1 \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= 2\pi \cdot \left.\frac{\rho^3}{3}\right|_0^1 \int_0^{\frac{\pi}{4}} \sin\phi \; d\phi$$