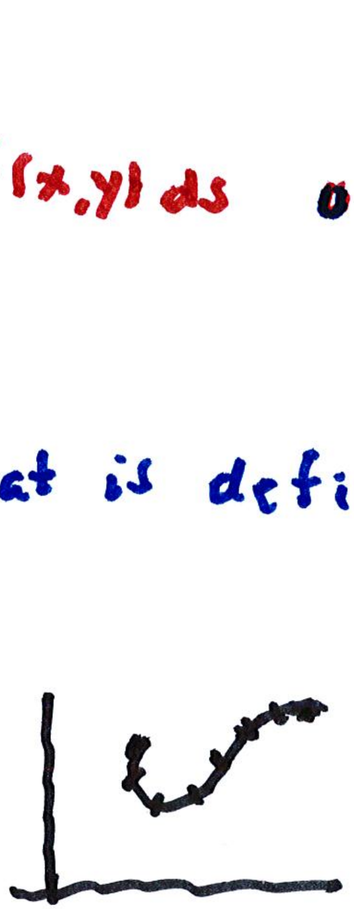
Suppose we are given a curve C defined by $\{x(t), y(t)\}$ for $a \le t \le b$.

We want to define the integral

$\int_C f(x,y)\,ds$ of a function $f(x,y)$

that is defined for all (x,y) on C.



As usual, we partition the curve

C by $a = t_0 < \dots t_{i-1} < t_i \dots < t_n$

The path from

$(x(t_{i-1}), y(t_{i-1}))$ to $(x(t_i), y(t_i))$

can be approximated by a

segment of length $\sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$

As usual, we partition the curve by defining points

$(x_i, y_i) = (x(t_i), y(t_i))$,

where $a < \dots < t_{i-1} < t_i < \dots t_n = b$.

The i-th segment has length approximately $\approx$

$$\ell_i = |\langle x'(t_i), y'(t_i)\rangle| \Delta t.$$

To define $\int_C f(x,y)\,ds$,

we multiply ℓ_i by $f(x_i, y_i)$

Thus, we obtain

$$\int f(x,y)\,ds \approx \sum_{i=1}^{n} f(x_i, y_i) \cdot \underbrace{|\langle x'(t_i), y'(t_i)\rangle|\,\Delta t}$$

$$\vec{r}(t) = \langle 1,2\rangle + t\langle 3,6\rangle, \quad 0 \le t \le 1.$$

$$\therefore\ x = 1 + 3t, \quad y = 2 + 6t$$

$$\langle x', y'\rangle = \langle 3, 6\rangle$$

$$|r'(t)| = |\langle 3,6\rangle| = \sqrt{45}$$

$$\int_0^1 x^2\, y\, |r'(t)|\, dt$$

$$= \int_0^1 (1 + 6t + 9t^2)\sqrt{45}\, dt$$

$$= \int_0^1 \sqrt{45}\,\left(t + 3t^2 + 3t^3 \right) dt \Big|_0^1$$

$$= \sqrt{45}\,\left(1\,t + t^3 + \frac{3}{4} \right) \Big|_0^1 = \underline{\underline{7\sqrt{45}}}$$

Ex. Compute $\displaystyle\int_{C_1} 2x \, ds + \int_{C_2} x^2$

where C_1 = parabolic arc $y = x^2$

from $(0,0)$ to $(1,1)$ and

where C_2 is the straight path from

$(1,1)$ to $(0,1)$.

$\vec{r}_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1$

and $\vec{r}_2(t) = (0,1) - t(1,0) \quad 0 \le t \le 1.$

$= (-t, 1)$

$|\vec{r}_1'| = \sqrt{1+4t^2}$

$$\int_{C_1} 2t \quad \sqrt{1+4t^2}\, dt$$

$$= \frac{1}{4}\int_0^1 \sqrt{1+4t^2} \cdot 8t\, dt$$

$$= \frac{1}{4} (1+4t^2)^{3/2} \cdot \frac{2}{3} \Big|_0^1$$

$$= \frac{1}{4} \frac{2}{3} \{ 5^{3/2} - 1^{3/2} \}$$

$|r'(t)| = 1$

$$\int_{C_2} x^2\, ds = -\int_0^1 (-t)^2 \cdot 1$$

$$= -\frac{t^3}{3}\Big|_0^1 = \frac{-1}{3}$$

Line integrals in space

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Compute $\displaystyle\int_C f(x, y, z)\, ds$

$$= \int_a^b f(x(t), y(t), z(t))\, |\vec{r}\,'(t)|\, dt.$$

We sometimes use the notation

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

to mean

$$\int_a^b P(x(t),y(t),z(t))\,x'(t)\,dt$$

$$+ \int_a^b Q(x(t),y(t),z(t))\,y'(t)\,dt$$

$$+ \int_a^b R(x(t),y(t),z(t))\,z'(t)\,dt$$

Ex. Evaluate $\int_C y\,dx - x\,dy + x\,dz$

where C is the straight path

from $(0,1,2)$ to $(1,2,-1)$.

C can be parameterized by

$(0,1,2) + t(1,1,-3)$

$x(t) = t, \quad y(t) = 1 + t, \quad z(t) = 2 - 3t$

$$\int_C = \int_0^1 (1-t)\cdot 1 - t(1) + t\cdot(-3)\, dt$$

$$= \int_0^1 -5t + 1 = -\frac{5t^2}{2} + t \Big|_0^1 = \frac{-3}{2}$$

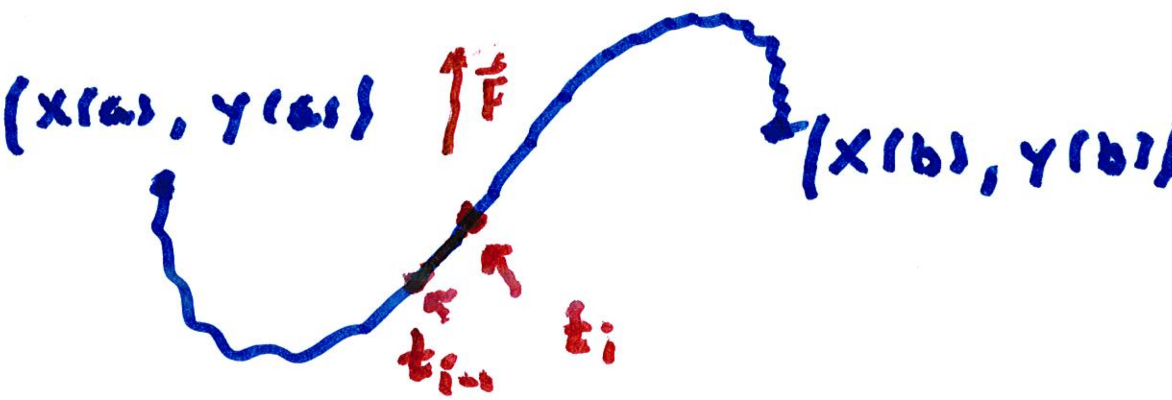
Recall that the work done

by a force (constant) is

$W = \vec{F} \cdot \vec{d}$.

Now suppose that the force

$\vec{F}(x,y,z)$ varies and also the path



$\Delta_i W = \vec{F}(x_i, y_i)$

$\cdot\ \vec{r}\,'(t_i)\ dt$

$$W = \int_C \langle F_1, F_2 \rangle \cdot \vec{r}\,'(t)\, dt$$

This is called the line integral of a vector field

Ex. Given a vector field

$\vec{F}(x,y) = y^2 \vec{i} + xy\vec{j}$, compute

the work W done by ~~F~~ a moving

particle that follows the

path $\vec{r}(t) = \langle \cos t, \sin t \rangle$ if $0 \le t \le \frac{\pi}{2}$

$\Rightarrow \vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$\therefore \vec{F}(t) \cdot \vec{r}'(t)$

$= \langle \cos^2 t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle$

$= -\cos^2 t \sin t + \cos^2 t \sin t = 0$

$\therefore W = \cancel{\int_0^{\pi/2} -2\cos^3 t \sin t \, dt} = 0$

$$= \frac{2\cos^3}{\cancel{3}} \Big|_0^{\frac{\pi}{2}} = -\frac{2}{\cancel{3}} = 0$$

\# 20. Evaluate $\int_C \vec{F} \cdot d\vec{r}$

where $\vec{F} = (x+y)\vec{i} + (y-z)\vec{j} + z^2\vec{k}$

and $\vec{r}(t) = t^2\vec{i} + t^3\vec{j} + t^2\vec{k}$, $0 \le t \le 1$

$\vec{r}'(t) = 2t\vec{i} + 3t^2\vec{j} + 2t\vec{k}$,

$$\vec{F}(r(t)) = (t^2 + t^3)\vec{i} + (t^3 - t^2)\vec{j} + t^4\vec{k}$$

$$\cancel{\vec{F} \cdot \vec{r}'(t) = 2t\vec{i} + 3t^2\vec{j} + 2t\vec{k}}$$

$$\vec{F} \cdot (\vec{r}'(t)) = 2t(t^2 + t^3) + (3t^5 - 3t^4)\vec{j}$$

$$+ 2t^5$$

$$= \frac{2}{4} + \frac{2t^5}{5} + \frac{3t^6}{6} - \frac{3t^5}{5} + \frac{t^6}{6}$$

$$= \frac{1}{2} - \frac{1}{5} + \frac{2}{3} = \frac{29}{30}$$