Compute $\int_C f(x,y)\,ds$ $\quad C = (x(t), y(t))$

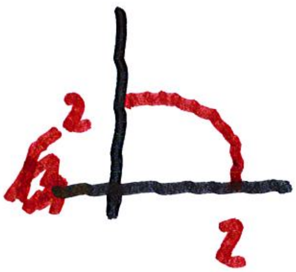
$a \le t \le b$

$$= \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

or $$= \int_a^b f(r(t))\, |\vec{r}\,'(t)|\, dt$$

Ex. Evaluate $\int_C xy\, ds$,

C = right ~~half~~ quarter circle $x^2 + y^2 = 4$



$$(x(t), y(t)) = (2\cos t, 2\sin t)$$

$$0 \le t \le \frac{\pi}{2}$$

$$(x'(t), y'(t))$$

$$= (-2\sin t, 2\cos t)$$

$$\int = \int_0^{\frac{\pi}{2}} 4\cos t \sin t \cdot 2 \, dt$$

$$= 8\int_0^{\frac{\pi}{2}} \sin t \cos t \, dt = \frac{8\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}$$

$$= 4$$

Evaluate $\displaystyle\int_C (x^2 + y^2 + z^2)\, ds$

for $x = t$, $y = \cos 2t$, $z = \sin 2t$,

$0 \le t \le 2\pi$.

$x' = 1$, $y' = -2\sin 2t$, $z' = 2\cos 2t$

$$|\vec{r}\,'(t)| = \sqrt{1 + 4\sin^2 2t + 4\cos^2 2t} \quad dt$$

$$= \sqrt{1+4} = \sqrt{5}$$

$$\int = \int_0^{2\pi} \left(t^2 + \cancel{4}\sin^2 2t + \cancel{4}\cos^2 2t\right)\sqrt{5}\, dt$$

$$= \int_0^{2\pi} \sqrt{5} \cdot \sqrt{5}\, dt$$

$$= 5 \cdot 2\pi = \underline{\underline{10\pi}}$$

Compute $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ $\quad C = \vec{r}(t)$, $a \le t \le b$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t),$$

or if $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$

and $C = \langle x(t), y(t) \rangle$, $a \le t \le b$

$$\int_a^b \big(P(x,y),\ Q(x,y)\big) \cdot \big(x'(t),\ y'(t)\big)$$

$$= \int_a^b \Big(P(\underset{x(t)}{x}, y)\, x'(t) + Q(x, \underset{y(t)}{y})\, y'(t)\Big)\, dt$$

Sometimes we have integrals

such as $\displaystyle\int_C P(x,y)\,dx + Q(x,y)\,dy$

$(x(t), y(t))$ $a \le t \le b$

Since $dx = x'(t)\,dt$, $dy = y'(t)\,dt$,

the integral becomes

$$\int_a^b \left(P(x,y)\,x'(t) + Q(x,y)\,y'(t) \right) dt$$

Compute $\displaystyle\int_C y\,dx + z\,dy + x\,dz$

for $C: \{\sqrt{t},\ t,\ t^2\}$, $1 \le t \le 4$

$$\{x', y', z'\} = \left\{\frac{1}{2\sqrt{t}}, 1, 2t\right\}$$

$$\therefore \int_1^4 t \cdot \frac{1}{2\sqrt{t}} + t^2 \cdot 1 + \sqrt{t} \cdot 2t \quad dt$$

$$= \int_1^4 \frac{\sqrt{t}}{2} + t^2 + 2t^{3/2} \quad dt$$

$$= \frac{1}{3} t^{3/2} + \frac{t^3}{3} + \frac{4}{5} t^{5/2} \Big]_1^4$$

$$= \left(\frac{1}{3} 8 + \frac{64}{3} + \frac{128}{5}\right) - \left(\frac{1}{3} + \frac{1}{3} + \frac{4}{5}\right)$$

$$= \frac{70}{3} + \frac{124}{5} = \frac{722}{15}$$

Ex. Compute $\displaystyle\int_C z^2\,dx + x^2\,dy + y^2\,dz$

if C = line segment from $\langle 1, 0, 0 \rangle$

to $\langle 4, 1, 2 \rangle$

$$\vec{r}(t) = \langle 1, 0, 0 \rangle + t\langle 3, 1, 2 \rangle, \quad 0 \le t \le 1$$

$$\therefore\ x(t) = 1 + 3t,\ y(t) = t,\ z(t) = 2t$$

$$x' = 3 \,, \quad y' = 1 \,, \quad z' = 2$$

$$\int_C = \int_0^1 (2t)^2 \cdot 3 + (1+3t)^2 + t^2 \cdot 2$$

$$= \int_0^1 12t^2 + 1 + 6t + 9t^2 + 2t^2 \, dt$$

$$= \int_0^1 23t^2 + 6t + 1 \, dt$$

$$= \frac{23}{3} + 3 + 1 = \frac{35}{3}$$

The definition of $\int_C \vec{F} \cdot d\vec{r}$ is

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds \, .$$

Note that

$$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r} \, ,$$

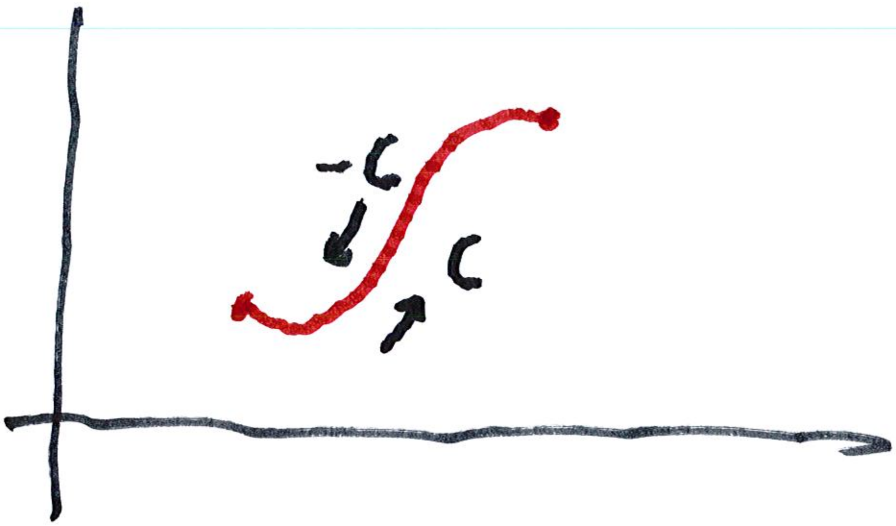
when the orientation of C

is reversed :

For example, in the previous problem if the curve is replaced by the line segment from $(4,1,2)$ to $(1,0,0)$, the answer would be $-\frac{35}{3}$.

Geometrically, $-C$ is the

same as C, but in the opposition,



$$C = \{ \vec{r}(t) ; \ a \le t \le b \}$$

$$-C = \{ \vec{r}(b - t(b-a)) ; \ a \le t \le b \}$$

Ex. Find the work done by the force field $\vec{F}(x,y) = x^2\vec{i} + xy\vec{j}$

if C_1 is the path $y = x^2$, $0 \le x \le 1$

and C_2 is the path $\langle 1, 1-t \rangle$.

$0 \le t \le 1$

C_1 $\quad \vec{r}_1(t) = \langle t, t^2 \rangle \quad r_1' = \langle 1, 2t \rangle$

$$\int_{C_1} (t^2)\cdot 1 + t\cdot t^2 \cdot 2t \, dt$$

$$= \int_0^1 t^2 + 2t^4 \, dt = \frac{t^3}{3} + \frac{2t^5}{5} \Big|_0^1$$

$$= \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$$

For C_2

$$\int_{C_2} 1^2 \cdot 0 + 1 \cdot (1-t)(-1)$$

$$= \int_0^1 (1-t)(-1) = \int_0^1 t-1 = \frac{t^2}{2} - t \Big|_0^1$$

$$= -\frac{1}{2}$$

$$\therefore \int_{C_1 + C_2} x^2\vec{i} + xy\vec{j} \cdot d\vec{r}$$

$$= \frac{11}{15} - \frac{1}{2} = \frac{7}{30}$$

Ex. Find the work done by the force field

$$\vec{F} = \langle x - y^2,\ y - z^2,\ z - x^2 \rangle$$

on a particle that moves along the line segment from

$(0,0,1)$ to $(2,1,0)$

$\vec{r}(t) = (0,0,1) + t(2,1,-1)$

$\therefore\ x(t) = 2t,\ y = t,\ z = 1-t.$

$\therefore\ x'(t) = 2,\ y' = 1,\ z' = -1$

$\int_C =$

$$= \int_0^1 (x-y^2)\cdot 2 + (y-z^2)\cdot 1 + (z-x^2)(-1)$$

$$= \int_0^1 (2x + y - z) + (+x^2 - 2y^2 - z^2)$$

$$= \int_0^1 2\cdot 2t + t - (1-t)$$

$$+ (2t) - 2t^2 - (1-t)^2$$

$$= \Big| \; 2t^2 + \frac{t}{2} - t + \frac{t^3}{2} + t^2 - \frac{2t^3}{3} - \frac{t^3}{3}$$