16.3 Fundamental Thm. for Line Integrals.

Given a function $F(x)$, $a \le x \le b$,

the Fundamental Thm. of Calculus

says:

$$\int_a^b F'(x) = F(b) - F(a).$$

For vector fields, there is

a similar statement,

Given a function $f(x,y)$

or also $f(x,y,z)$, we can define

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$\left(\text{or also } \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \right)$$

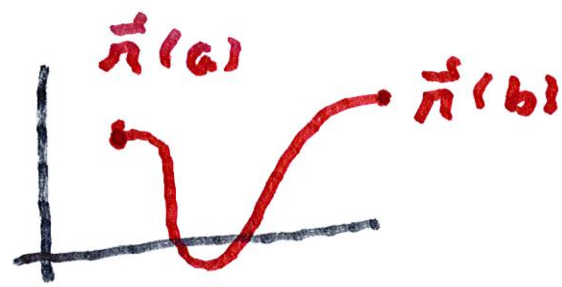
called the gradient of f.

Thm. Suppose $f(x,y)$ is C^1

near the curve $\vec{r}(t)$, for $a \le t \le b$.

Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(b)).$$



If $\vec{r}(a) = \langle x_1, y_1 \rangle$

and $\vec{r}(b) = \langle x_2, y_2 \rangle$

In 3 variables if

$\vec{r}(a) = (x_1, y_1, z_1)$

and $\vec{r}(b) = (x_2, y_2, z_2)$, then

$$\int_C \nabla f \cdot d\vec{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Pf. $$\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f_{(r(t))} \cdot \vec{r}\,'(t)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(x(t), y(t), z(t))\, dt$$

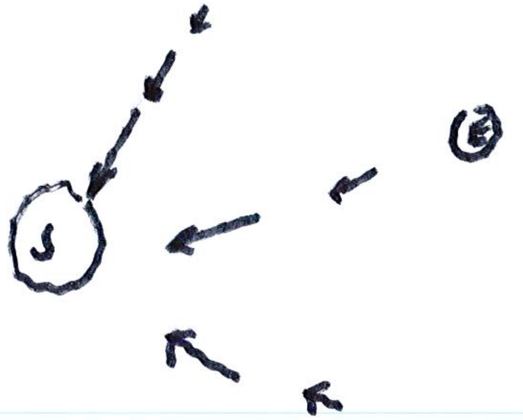
$$= f(x(t), y(t), z(t)) \Big|_a^b$$

$$= f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Ex. Gravitational field

The gravitational field is

$$\vec{F}_g(\vec{x}) = -\frac{mMG\vec{x}}{|\vec{x}|^3}$$



M = mass of sun

m = mass of earth

Set $f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$

Then $\nabla f = \overline{F}_g$

If an object moves from

$(3, 4, 12)$ to $(2, 2, 0)$

Thus, if $\vec{F} = \nabla f$, then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

is **independent of path.**

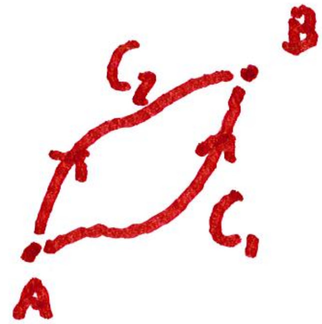
A curve C is closed if

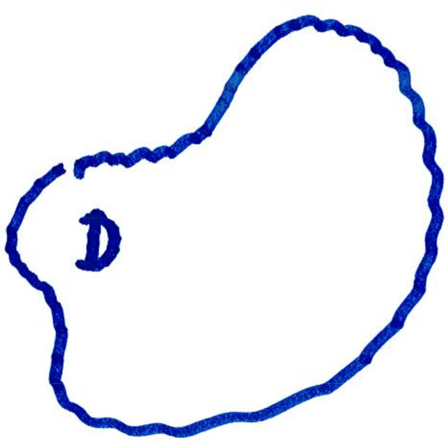
$$\vec{r}(b) = \vec{r}(a)$$

If $\int \vec{F} \cdot d\vec{r}$ is independent of path,

$$= \int_{C_1} \vec{F}\cdot d\vec{r} \;-\; \int_{C_2} \vec{F}\cdot d\vec{r} \;=\; 0, \text{ i.e}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot dr$$





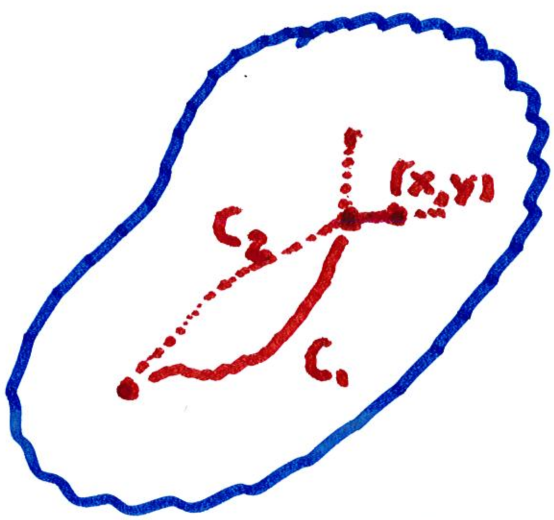
If $\int_C \vec{F}\cdot dr$ is

independent of path, then

there is a function f in D

so that $\nabla f = \vec{F}$

Write $\vec{F} = P\vec{i} + Q\vec{j}$



$\Rightarrow$ Set $f(x,y) = \int_{C_1} P$

Or set $f(x,y) = \int_{C_2} Q\,dy$

How do we know if $\vec{F} = \nabla f$

for some f?

If $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$

then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ If $\nabla f = P(x,y)\vec{i} + Q(x,y)\vec{j}$,

then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$