16.6 Parametric Surfaces

\# 38. p. 1098

Maxwell's Equations. Let $\vec{E}(x_1, x_2, x_3, t)$ and $\vec{H}(x_1, x_2, x_3, t)$ denote the electrical and magnetic field at (x_1, x_2, x_3, t)

Maxwell showed that $\vec{E}$ and $\vec{H}$ satisfy

$$\text{div}\,\vec{E} = \vec{0} \qquad \text{div}\,\vec{H} = 0$$

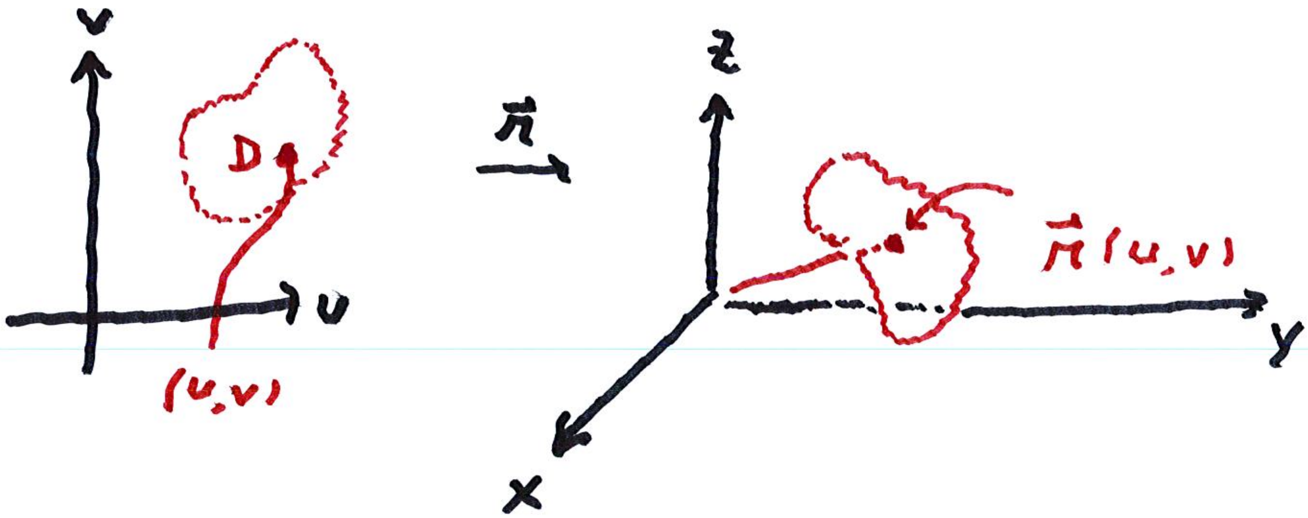
$$\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial H}{\partial t} \qquad \text{curl}\,\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

He showed that light is an electro-magnetic wave with c = speed of light.

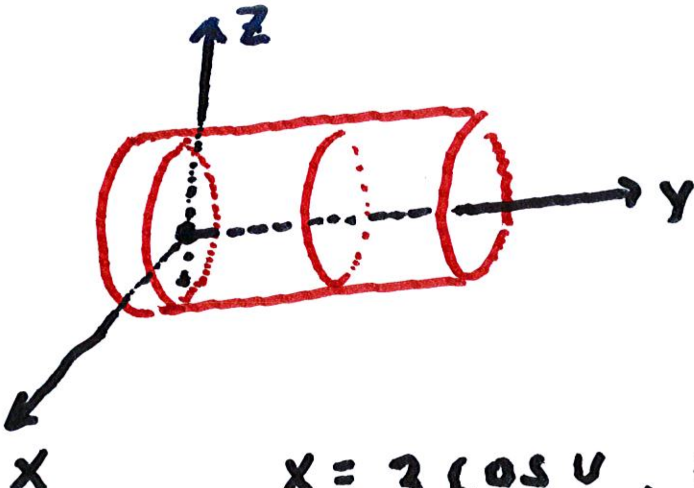
16.6 We can describe a surface by

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

As u and v change they describe a surface in $\mathbb{R}^3$.



Ex. A cylinder of radius 3 :



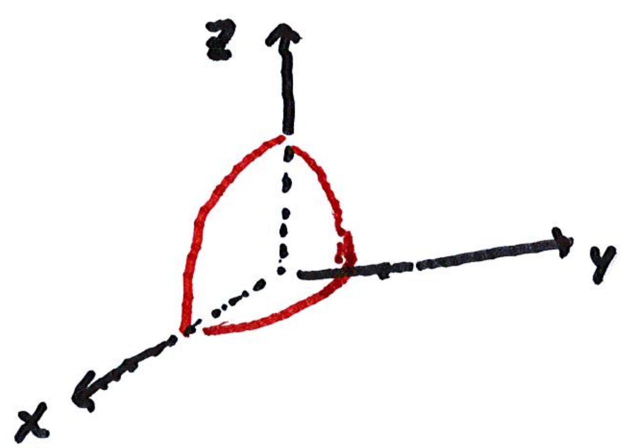
$x = 3\cos u\ ,\ z = 3\sin u\ ,\ y = v$

or:

$$\vec{r}(u,v) = 3\cos u\,\vec{i} + v\,\vec{j} + 3\sin u\,\vec{k}$$

Spherical Coord:

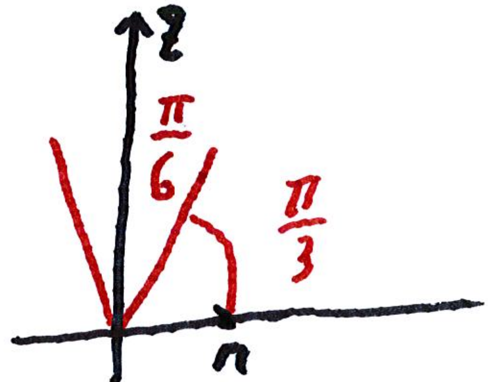
$$\left.\begin{aligned} x(\theta,\phi) &= 4\cos\theta\sin\phi \\ y(\theta,\phi) &= 4\sin\theta\sin\phi \\ z(\theta,\phi) &= 4\cos\phi \end{aligned}\right\} \quad \begin{aligned} &0<\theta<\frac{\pi}{2} \\ &0<\phi<\frac{\pi}{2} \end{aligned}$$



A sphere of radius 4 in first octant

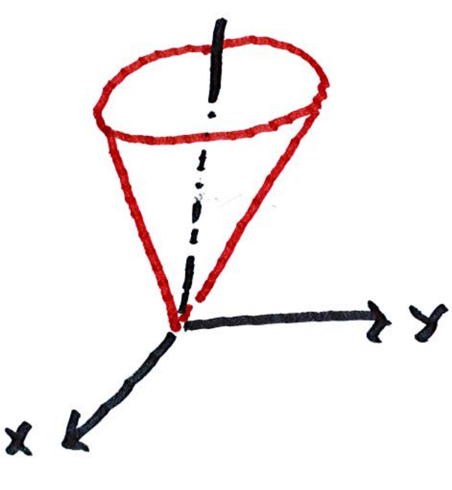
Ex. Give a parameterization of the top half of the cone with angle $\frac{\pi}{6}$ from the central axis

First when $x = 0$



$z = \sqrt{3}\, r$

Now with $x \neq 0$, $z = \sqrt{3}\sqrt{x^2 + y^2}$

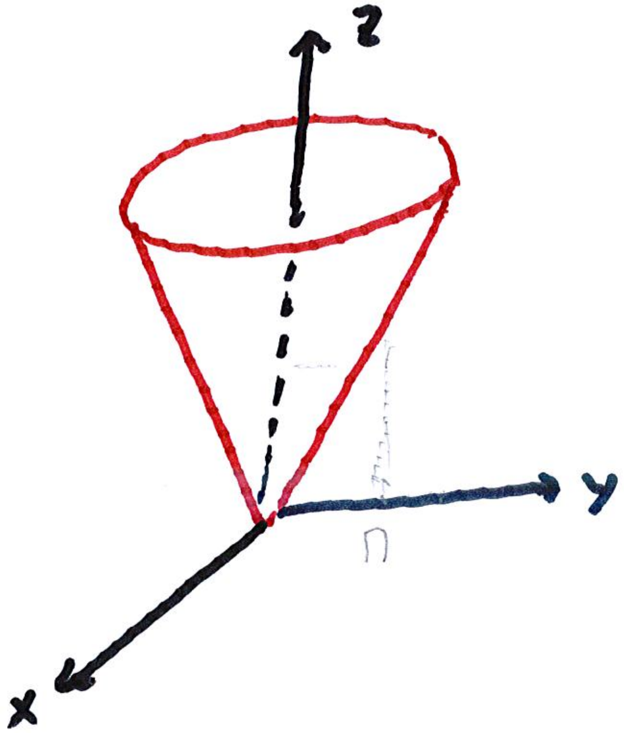


With parameters r and θ

with $0 \leq \theta < 2\pi$, $x = r\cos\theta$

$0 \leq r \leq 10$ $\quad y = r\sin\theta$

$z = \sqrt{3}\, r$



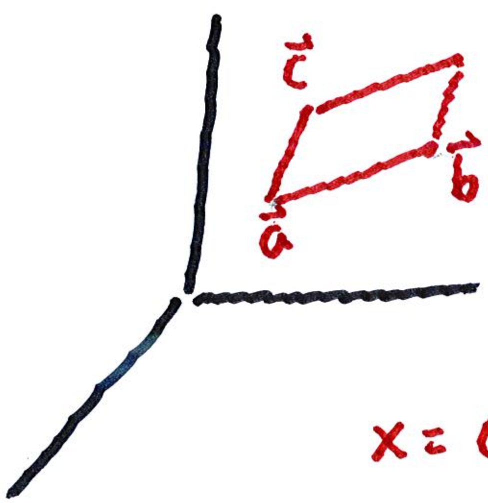
Ex. Find a parameterization of

the parallelogram passing through

$\vec{a}$, $\vec{b}$, and $\vec{c}$:

$$(u,v) \to \vec{a} + u(\vec{b} - \vec{a}) + v(\vec{c} - \vec{a})$$

for $0 \le u \le 1$ and $0 \le v \le 1$



$$x = a_1 + u(b_1 - a_1) + v(c_1 - a_1)$$

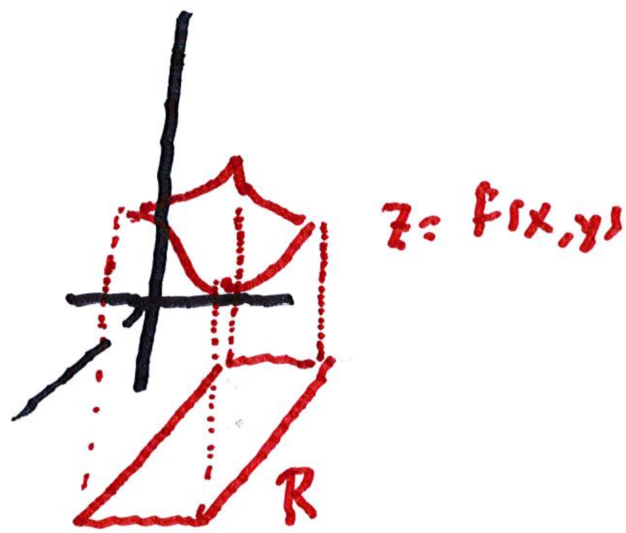
$$y = a_2 + u(b_2 - a_2) + v(c_2 - a_2)$$

$$z = a_3 + u(b_3 - a_3) + v(c_3 - a_3)$$

Write a parameterization

of the graph of $f(x,y)$ for

$(x,y) \in D$

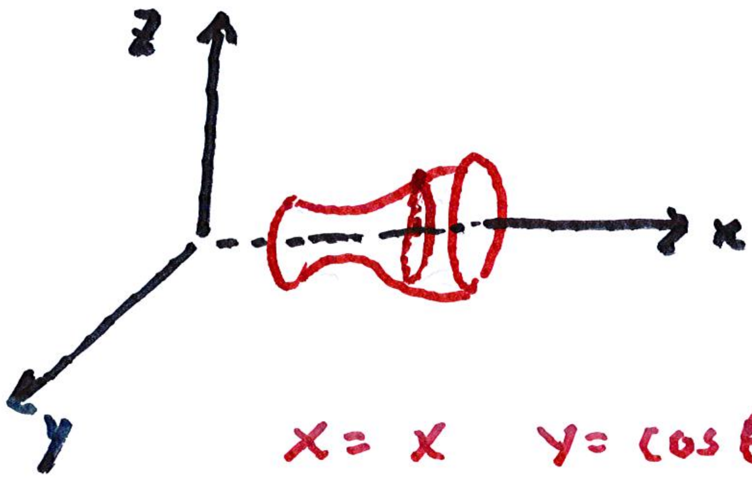


$$(u,v) \in R \xrightarrow{\vec{r}} (u,\, v,\, f(u,v))$$

Surfaces of Revoution

Revolve $y = f(x)$, $a \le x \le b$

about the x-axis



$x = x \quad y = \cos\theta\, f(x) \quad z = \sin\theta\, f(x)$

when $\theta = 0$, we get the usual

curve $y = f(x)$.

when $\theta = \frac{\pi}{2}$, we get $y = f(x)$

$\therefore (x,\theta) \to (x, \quad = \left(x, \cos\theta\, f(x), \sin\theta\, f(x)\right)$

for $0 \le x \le 4$

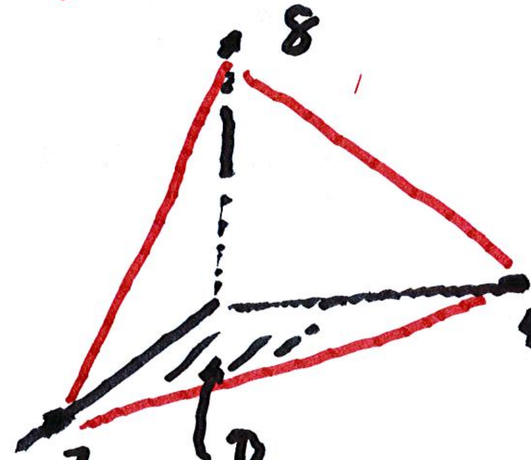
and $0 \le \theta \le 2\pi$

Find a pamameterization

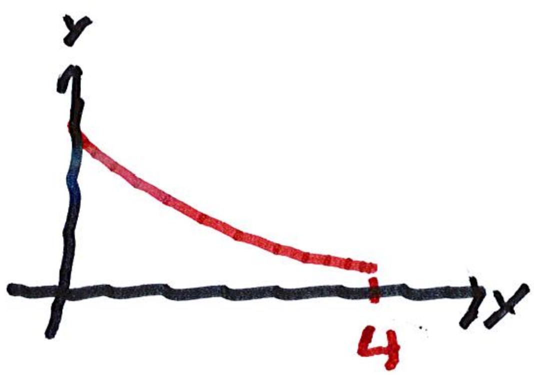
of the triangle in the first

octant generated by

$4x + 2y + z = 8$



Ex. Find a parameterization of the surface obtained by rotating the curve $y = e^{-x}$ for $0 \le x \le 4$ about the x-axis

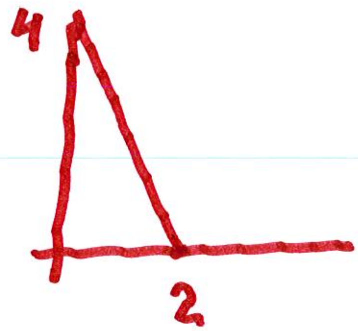


$x = x$

$y = e^{-x} \cos \theta$

$z = e^{-x} \sin \theta$

Let D = base in the plane $z=0$



$$4x + 2y = 8$$

or $$2x + y = 4$$

$$\boxed{\begin{array}{c} 0 < y < 4 - 2x \\ x > 0,\ y > 0 \end{array}}$$

D

$$0 < z < 8 - 4x - 2y$$

We study how $\vec{r}(u,v)$ when we only let u change

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\,\vec{k}$$

and then only let v change:

$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\,\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\,\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\,\vec{k}$$

$\therefore\ \vec{r}_u$ and $\vec{r}_v$ are tangent to

S. The parallelogram with Δu and Δv

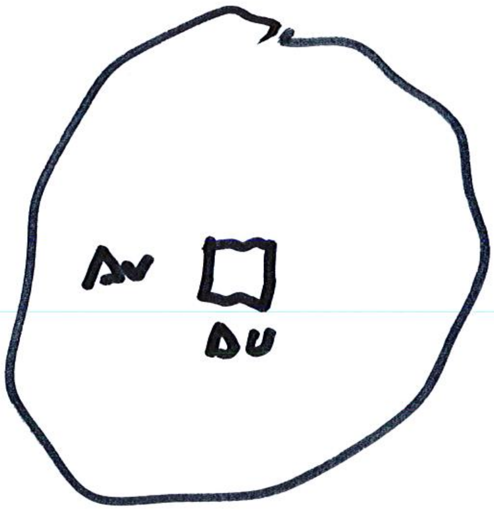
generates

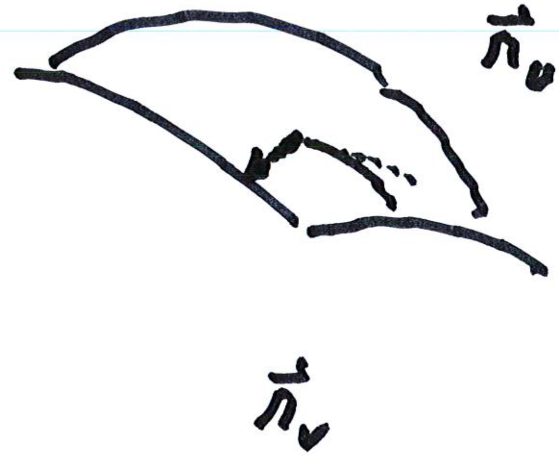
$$\bullet\ \{r_u \Delta u,\ | r_v \Delta v\}$$

$$= \Delta u \cdot \Delta v \cdot \left| \vec{r}_u \times \vec{r}_v \right|.$$

The normal to the surface is

$$\vec{r}_u \times \vec{r}_v$$





We obtain 2 short segments

$(\Delta u)\vec{r}_u$, $\Delta v\, \vec{r}_v$

Area of $\Delta u \Delta v$ image

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \left\| \begin{vmatrix} i & j & k \\ & \vec{r}_u \Delta u & \\ & \vec{r}_v \Delta v & \end{vmatrix} \right\|$$

$$= |\Delta u|\,|\Delta v|\,\left| \vec{r}_u \times \vec{r}_v \right|$$ = infinitesimal amount of area gen by Δu, Δv

$$\therefore \text{ Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$\uparrow$ C

Area of plane above par. dom

$$= \iint_D |\vec{r}_u \times \vec{r}_v| = C \cdot A(D)$$