# 3.7 Infinite Series

To define an infinite series of the form $\sum_{n=1}^{\infty} x_n$, we define a sequence

$$S_N = \sum_{n=1}^{N} x_n \quad \text{for } N = 1, 2, \ldots$$

If the sequence $S_N$ converges to $S$, we say the series converges and we write $\sum_{n=1}^{\infty} x_n = S$.

Ex. Consider the series

$$\sum_{n=0}^{\infty} r^n .$$ If $r \neq 0$, then

$$S_N = \sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r} .$$

When $|r| < 1$, $S_N$ converges to $\frac{1}{1-r}$. Hence

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} .$$

Telescoping Series.

Ex. Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

converges and find its value.

Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\therefore S_N = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$$

$$+ \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

By cancellation:

$$S_N = \frac{1}{1} - \frac{1}{N+1} \longrightarrow 1 \quad \text{as } N \to \infty.$$

Suppose $\sum x_n$ converges.

Since $S_N \to S$ as $N \to \infty$,

given $\varepsilon > 0$, there is a $K$,

so that if $l \geq K$, then

$|S_l - S| < \varepsilon$.

But if $N \geq K+1$, then $N-1 \geq K$,

so $|S_{N-1} - S| < \varepsilon$.

Hence $S_N$ and $S_{N-1}$ both

converge to $S$.

If we write $S_N - S_{N-1} = x_N$,

then by letting $N \to \infty$, we

get $S - S = \lim_{N \to \infty} x_N$.

It follows that if $\sum_{n=1}^{\infty} x_n$ converges,

then $\underline{\lim x_n = 0}$

Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2-1}}{3n+5}$ converge?

Compute $\displaystyle\lim \frac{\sqrt{2n^2-1}}{3n+5}$

$$= \frac{n\sqrt{2-\frac{1}{n^2}}}{n\left(3+\frac{5}{n}\right)} \to \frac{\sqrt{2}}{3} \neq 0$$

as $n \to \infty$

Since $(X_n)$ does NOT approach 0,

it follows that the series

diverges.

Ex. Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Look at

$$S_{2^k} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$+ \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$\vdots$$

$$+ \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{k-1}}{2^k}$$

$$= 1 + \frac{k}{2} \longrightarrow \infty \text{ as } k \to \infty.$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Ex. For $p>1$, we want to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

We modify the above method:

$$S_{2^{k+1}-1} = 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots \frac{1}{7^p}\right)$$

$$+ \cdots + \left(\frac{1}{2^{kp}} + \frac{1}{(2^k+1)^p} + \cdots \frac{1}{(2^{k+1}-1)^p}\right)$$

$$S_{2^{k+1}-1} \leq 1 + \frac{2}{2^p} + \frac{4}{4^p} \cdots + \frac{2^k}{2^{kp}}$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \left(\frac{1}{2^{p-1}}\right)^3$$

$$\cdots + \left(\frac{1}{2^{p-1}}\right)^k$$

If we set $r = \frac{1}{2^{p-1}}$, then $0 < r < 1$.

$$S_{2^{k+1}-1} \leq 1 + r + r^2 + \cdots + r^k.$$

$$= \frac{1 - r^{k+1}}{1 - r} < \frac{1}{1-r}.$$

In general, if we let $r = \frac{1}{2^{p-1}}$

we have $0 < r < 1$, and

$0 < S_{k_j} < 1 + r + r^2 + \dots + r^{j-1}$.

The whole infinite series

is $\sum_{j=1}^{\infty} r^{j-1} = \frac{1}{1-r}$

We conclude that the

p-series is convergent

when $p > 1$.

The last conclusion actually follows from the following:

Comparison Test. Suppose that $(x_n)$ and $(y_n)$ satisfy

$$0 \le x_n \le y_n, \quad \text{if } n \ge K.$$

Then

(a) The convergence of $\sum y_n$ implies the convergence of $\sum x_n$

(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

For (a). Let $S_N$ be the partial sum of $\sum_{n=1}^{\infty} x_n$ and let $T_N$ be the partial sum of $\sum_{n=1}^{\infty} y_n$.

Clearly $S_N \leq T_N$. Since $T_N$ is bounded for all n, it follows that $\sum_{n=K}^{\infty} x_n \leq \sum_{n=K}^{\infty} y_n$.

We know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$

converges if $p > 1$

and it diverges if $p < 1$.

When $p < 1$, then clearly the Comparison Test implies that $\frac{1}{n^p} \geq \frac{1}{n}$ when $\underline{\underline{p < \frac{1}{2}}}$.

Hence (b) of the Comparison implies that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

Ex. Determine the convergence

of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n^2-1}}{3n^3-4}$

The n-th term is $\sim \dfrac{n}{n^3}$.

But if the denominator were

$3n^3+4$, we could use the

usual comparison test.

$$\frac{\sqrt{2n^2-1}}{3n^3+4} \leq \frac{\sqrt{2n^2}}{3n^3} = \frac{\sqrt{2}}{3}\,\frac{1}{n^2}$$

It's better to use the Limit Comparison Test.

Suppose $(x_n)$ and $(y_n)$ are both positive and satisfy

$$r = \lim \left( x_n / y_n \right) \neq 0 .$$

Then $\sum x_n$ converges if and only if $\sum y_n$ converges.

Proof $\varepsilon = \frac{\Omega}{2}$. Then there is a whole number $K$ so that if $n \geq K$, then

$$\Omega - \varepsilon < \frac{x_n}{y_n} < \Omega + \varepsilon .$$

or
$$\frac{\Omega}{2} < \frac{x_n}{y_n} < \frac{3\Omega}{2} .$$

Then $x_n < \frac{3\Omega}{2} y_n$

and $y_n < \frac{2}{\Omega} x_n$

$\Bigg\}$ conv. of one $\Rightarrow$ conv. of other

$x_n$

For $\sum \frac{\sqrt{2n^2-1}}{3n^3-4}$,

Set $y_n = \frac{\sqrt{n^2}}{n^3} = \frac{1}{n^2}$.

Must show

$$\lim \frac{\frac{\sqrt{2n^2-1}}{3n^3-4}}{\frac{1}{n^2}} = \frac{n^2 \cdot n\sqrt{2 - 1/n^2}}{n^3(3 - 4/n^3)}$$

$\rightarrow \frac{\sqrt{2}}{3}$ as $n \to \infty$. Since

$\sum \frac{1}{n^2}$ conv., so does $\sum x_n$

The Limit Comp. Test does not apply to $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)}$.

There's no way to simplify $x_n$.

The integral test is best here.

$$\int_3^{\infty} \frac{1}{x \ln x}\,dx = \ln(\ln x)\Big]_3^{\infty} = \infty - \ln(\ln 3)$$

Also L'Hopital's Rule works, but we'll learn about these later.

# Alternating Series.

If $\sum_{n=1}^{\infty} (-1)^n a_n$ is a series

with $a_1 > a_2 > \cdots > a_n > 0$.

Then the series converges

if and only if $\lim_{n\to\infty} a_n = 0$.

The only if statement follows

since converg $\sum_{n=0}^{\infty} a_n$ implies $\lim_{n\to\infty} a_n = 0$

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Ex. Show that $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

In fact $\frac{1}{\sqrt{n}}$ is decreasing,

so $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges.

Note that $\left| (-1)^n \frac{1}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}}$,

so $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.