Def'n. Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$.

We say that f is uniformly continuous on A if for every $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that if $x_1, x_2 \in A$ are any numbers satisfying $|x_1 - x_2| < \delta(\varepsilon)$, then $|f(x_1) - f(x_2)| < \varepsilon$.

The point is that if we want to guarantee that $|f(x_1) - f(x_2)|$, it suffices to choose δ sufficiently small, say $|x_1 - x_2| < \delta(\varepsilon)$.

Thm. If $I = [a, b]$ is a closed bounded interval, and f is continuous on I, then f is uniformly continuous on I.

Pf. If f is not uniformly continuous on I, then there is a number $\varepsilon_0 > 0$, such that for any number $\delta > 0$, there are numbers $u = u(\delta)$ and $v = v(\delta)$ such that

$$|u(\delta) - v(\delta)| < \delta,$$ but

that $$|f(u(\delta)) - f(v(\delta))| \geq \varepsilon_0$$

In fact, for every $n \in \mathbb{N}$, there are numbers u_n and v_n in I such that $|u_n - v_n| < \frac{1}{n}$ but that $|f(u_n) - f(v_n)| \geq \varepsilon_0$.

Since I is bounded, the Bolzano-Weierstrass Thm implies that the sequence (u_n) has a subsequence

(u_{n_k}) that converges to a number x in $\mathbb{R}$.

Since $a \le u_{n_k} \le b$ for all $k = 1, 2, \ldots,$ it follows that

$$x = \lim_{k \to \infty} u_{n_k} \text{ also is in } [a, b].$$

Note that

$$|v_{n_k} - x| \le |v_{n_k} - u_{n_k}| + |u_{n_k} - x|$$

We know $|v_n - u_n| < \frac{1}{n} \to 0$

In particular, $\lim_{k \to \infty} |v_{n_k} - u_{n_k}|$

approaches 0. In addition,

we know that $u_{n_k} - x$ also

approaches 0. We conclude

that $\lim_{k \to \infty} v_{n_k} = x$.

It is clear that both

u_{n_k} and v_{n_k} approach x.

Since f is continuous at x, both $f(u_{n_k})$ and $f(v_{n_k})$ converge to $f(x)$. But this is impossible since

$$|f(u_n) - f(v_n)| \geq \varepsilon_0.$$

Thus, our assumption that f is not uniformly continuous implies that f is not continuous at some point x in I.

Consequently, if f is continuous at every point of I, then f is uniformly continuous on I.

Lipschitz Functions.

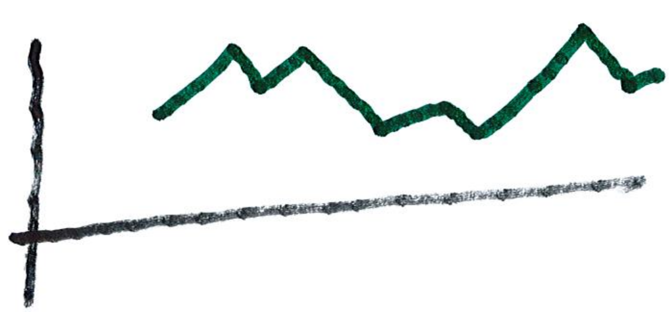
Def'n. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. If there exists a constant $K > 0$ such that $|f(x) - f(u)| \leq K|x-u|$,

(1)

for all $x, u \in A$, then f is said to be a Lipschitz function on A.

Geometrically, the Lipschitz condition can be written as

$$\left| \frac{f(x) - f(u)}{x - u} \right| \leq K$$



Thus, the slopes of all the segments joining two points on the graph of $y = f(x)$ are bounded by a constant K.

Thm. If $f: A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous

Pf If (1) is true, then given $\varepsilon > 0$, we can take $\delta = \varepsilon / K$. If $x, u \in A$ satisfy $|x-u| < \delta$, then

$$|f(x) - f(u)| \le K|x-u|$$

$$\le K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Ex. The function $g(x) = \sqrt{x}$

is continuous on $[0,1]$,

but it is not Lipschitz.

because if

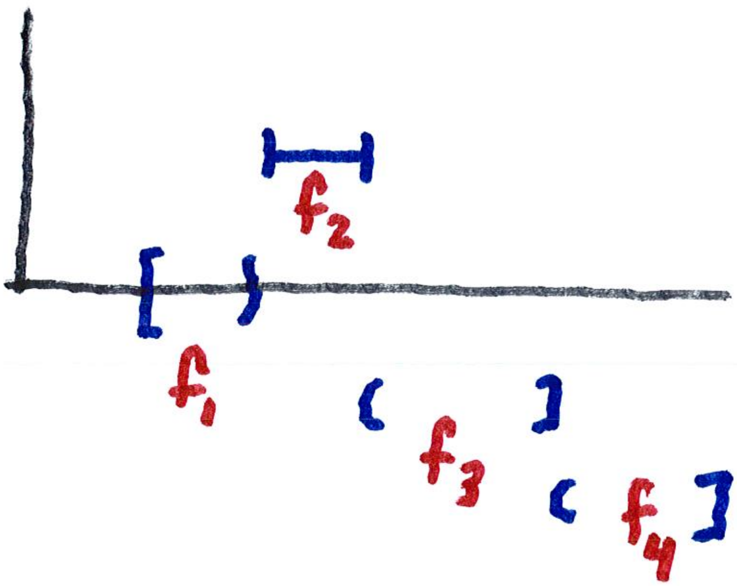
$g(x) - g(0) \leq K(x-0) = Kx$,

then $\sqrt{x} \leq Kx$ for all $x \in [0,1]$.

Thus $1 \leq K\sqrt{x}$. But this

cannot happen if x is small in $[0,1]$

Def'n. Let $I \subseteq \mathbb{R}$ be an interval and let $s : I \to \mathbb{R}$. Then s is called a step function if it has only a finite number of values. Moreover, on each interval, the step function takes on only one value in the interior of each interval.



Thm. Let $I = [a, b]$ be a closed bounded interval, and let $f: I \longrightarrow \mathbb{R}$ be continuous on I. If $\varepsilon > 0$, then there exists a step function $s : \bar{I} \to \mathbb{R}$

such that $|f(x) - s(x)| < \varepsilon$

for all $x \in \bar{I}$.

Pf. The function f is

uniformly continuous, so

given $\varepsilon > 0$, there is a

number $\delta(\varepsilon)$ such that

if $x, y \in \bar{I}$ and $|x-y| \leq \delta$, then $|f(x) - f(y)| < \varepsilon$.

Let $I = [a, b]$ and let m be sufficiently large so that $h = (b-a)/m < \delta(\varepsilon)$

Now we divide $[a, b]$ into m disjoint intervals of length h.

$$a = x_0 < x_1 \dots < x_{m-1} < x_m = b.$$

where $x_i - x_{i-1} = h = \dfrac{b-a}{m}$.

Now define

$s_\varepsilon(x) = f(a + kh)$, for all

$x \in I_k$, $k = 1, \ldots, m$,

so s_ε is constant on each

interval (The value of s_ε

on I_k is the value of f

at the right endpoint of I_k

Hence, if $x \in I_k$, then

$$|f(x) - s_\varepsilon(x)| = |f(x) - f(a + kh)|$$

$$< \varepsilon .$$

Hence $|f(x) - s_\varepsilon(x)| < \varepsilon$

for all $x \in I$.