Let's try a different proof of Taylor's Theorem:

Suppose that

(1) f is continuous in the closed interval determined by a and x;

(2) $f^{(n)}(a)$ exists;

(3) $f^{(n+1)}$ exists in the interior of I.

Then $f(x) = P_n(x) + R_n(x)$,

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}$ and

$$R_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!} (x-a)^{n+1},$$

where η is some point

in the interior of I.

Proof To simplify the notation,

we shall abbreviate $R_n(t)$

to $R(t)$.

The essential facts about

$R(t)$ are:

(a) $R(a) = R'(a) = \cdots = R^{(n)}(a)$

(b) $R^{(n+1)}(t) \equiv f^{(n+1)}(t)$.

The latter fact comes from the fact that

$$R(t) = f(t) - P_n(t)$$

and that $P_n(t)$ is a polynomial of degree at most n.

We introduce a function φ as follows:

$$\varphi(t) = R(t) - K(t-a)^{n+1},$$

where K is a constant which we choose so that $\varphi(x) = 0$.

Thus,

$$0 = \varphi(x) = R(x) - K(x-a)^{n+1}$$

Thus,

$$K = \frac{R(x)}{(x-a)^{n+1}} \qquad (4)$$

Keep in mind that K and x are constants and that we fix them throughout the entire argument.) The function φ has these properties:

(a') $\varphi(x) = \varphi(a) = \varphi'(a) = \dots = \varphi^{(n)}(a) = 0;$

(b') $\varphi^{(n+1)}(t) = f^{(n+1)} - (n+1)!\,K$

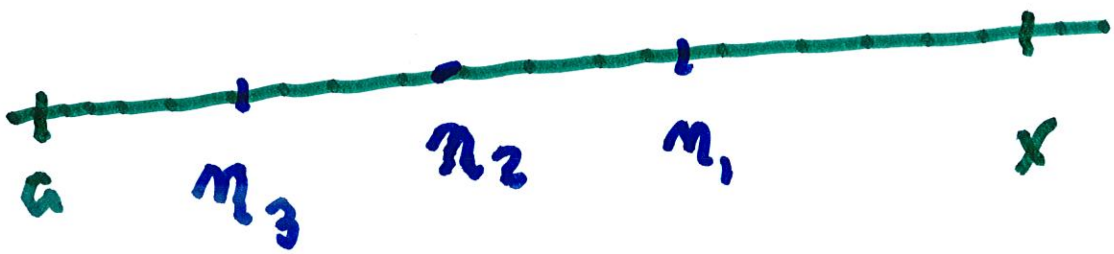
We now apply Rolle's theorem to φ and its derivatives.

Since $\varphi(x) = \varphi(a) = 0$,

Rolle's Theorem implies that

there is a number η_1 between

a and x such that $\varphi'(\eta_1) = 0$.



Continuing, since $\varphi'(a) = \varphi'(\eta_1) = 0$

there is an η_2 between

a and η_1 and that satisfies

$\varphi''(\eta_2) = 0.$

In this way we obtain numbers

$$x > \eta_1 > \eta_2 > \cdots > \eta_n > a$$

satisfying

$$0 = \varphi(x) = \varphi'(\eta_1) = \cdots \varphi^{(n+1)}(\eta_{n+1}) = 0.$$

For ease of notation, we set

$\eta = \eta_{n+1}$. We observe from

(b') that

$$0 = \varphi^{(n+1)}(\eta) = f^{(n+1)}(\eta) - (n+1)!\,K$$

which we can write as

$$K = \frac{f^{(n+1)}(\eta)}{(n+1)!}$$

Combining this with (4),

we get

$$R_n(x) = K(x-a)^{n+1}$$
$$= \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-a)^{n+1},$$

which is the Lagrange Remainder

Formula.

This completes the proof of Taylor's Theorem with Lagrange's Error Formula.

Corollary. Let I be an interval [a, x] and that

$$M_{n+1} = \sup\left\{ |f^{(n+1)}(t)| ;\ t \in [a, x] \right\}$$

It is obvious that

$$|f^{(n+1)}(n)| \le M_{n+1}.$$

Thus we get

$$|R_n(x)| \le \frac{M_{n+1}\,|x-a|^{n+1}}{(n+1)!}$$

This complete's the proof of Taylor's Theorem with Lagrange's Remainder.

Corollary. Let I be an interval containg a and suppose that

$$|f^{(n+1)}(x)| \leq M \quad \text{for all } x \in I,$$

where M is a constant. Then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Ex. Consider the function $f(x) = e^x$. Note that for any fixed $d > 0$,

$$\sup\{ f^{(n)}(x) ; |x| \leq d \} = e^d.$$

Thus $M = e^d$. If we write

$$P_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

By the error estimate,

$$\left| f(x) - \sum_{n=0}^{N} \frac{x^n}{n!} \right|$$

$$\leq \frac{e^d |x|^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

We conclude that

$$\lim_{N\to\infty} |e^x - P_N(x)| = 0,$$

for all x with $|x| \leq e^d$.

Hence $e^x = \sum_{n=0}^{N} \frac{x^n}{n!}$.

Theorem for Higher-Derivative Test for Relative Extrema:

Suppose that

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

and that $f^{(n)}(x_0) \neq 0$. Then

(1) f has a strict relative minimum at x_0 if n is even

(2) f has a strict relative maximum at x_0 if n is odd.