Recall that R_h is a rectangle in $\mathbb{R}$ defined by

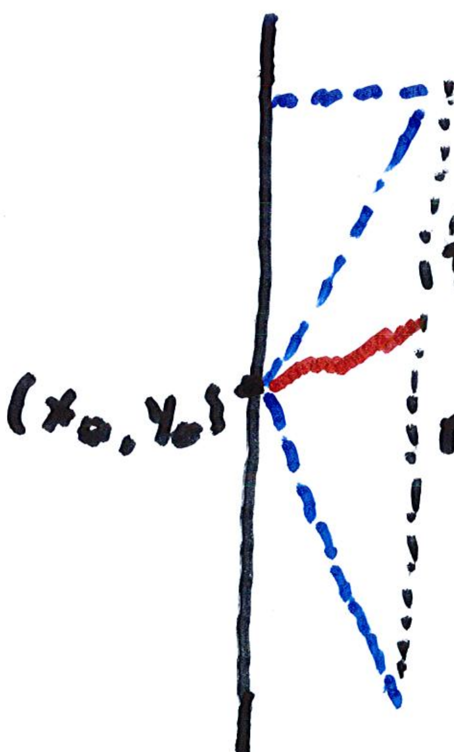
$$R_h = \left\{ (x,y) : \begin{array}{l} |x - x_0| \le h \\ |y - y_0| \le b \end{array} \right\}$$

We assume that $f(x,y)$ is continuous on R_h, and so there is a constant $M > 0$ such that $Mh \le b$. *

We also assume that f satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$$

for any points (x, y_1) and (x, y_2) in R_h.



If (*) is not true, we can replace h by a smaller one.

We want to show that there is a differentiable function $y(x)$ such that $y(x_0) = y_0$ and $y'(x) = f(x, y(x))$.

It is sufficient to show that there is a function $y(x)$ satisfying

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

Our first crude guess is

$y_0(x) = y_0$ for $x_0 \le x \le h$

Then we define $y_1(x)$ by

$$y_1(x) = y_0 + \int_{x_0}^{x} f\{t, y_0(t)\}\, dt.$$

for $x_0 \le x \le x_0 + h$. Then

we define

$$y_n(x) = y_0 + \int_{x_0}^{x} f\{t, y_{n-1}(t)\}\, dt$$

Lemma 1: If $|x-x_0| \le h$, then

$$|y_n(x)-y_{n-1}(x)| \le \frac{MK^{n-1}|x-x_0|^n}{n!}$$

$$\le \frac{MK^{n-1}h^n}{n!},$$

for $n = 1, 2, 3, \ldots$ We use induction.

Then $y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$

so $|y_1(x) - y_0| \le \left|\int_{x_0}^{x} f(t, y_0)\,dt\right|$

$$\le M|x-x_0|,$$

Assuming that

$$|y_{n-1}(x) - y_{n-2}(x)| \le \frac{MK^{n-2}|x - x_0|^{n-1}}{(n-1)!},$$

we must show that

$$|y_n(x) - y_{n-1}(x)| \le \frac{MK^{n-1}|x - x_0|^n}{n!}.$$

We must show this in the case when $x_0 \le x \le x_0 + h$.

By the Lipschitz Condition, we have

$$|y_n(x) - y_{n-1}(x)|$$

$$= \left| \int_{x_0}^{x} \left[f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)) \right] dt \right|$$

$$\leq \int_{x_0}^{x} \left| f(t, y_{n-1}(t)) - f(t, y_{n-2}(t)) \right| dt$$

$$\leq K \int_{x_0}^{x} \left| y_{n-1}(t) - y_{n-2}(t) \right| dt$$

Using hypothesis , we conclude

that

$$|y_n(x) - y_{n-1}(x)|$$

$$\leq \frac{MK^{n-1}}{(n-1)!} \int_{x_0}^{x} (t - x_0)^{n-1}\, dt,$$

or

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}}{n!} |x - x_0|^n.$$

When $x_0 - h \leq x \leq x_0$, a similar argument yields the same result. This completes the proof of the Lemma.

To utilize the lemma, we first compare the two infinite series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{MK^{n-1} h^n}{n!}$$

The second series is an absolutely convergent series. Moreover, by the lemma, the second series dominates the first series.

Here we make a brief digression:

Weierstrass M-Test. Suppose that $\{f_n\}$ is a sequence of real functions defined on a set A and that there is a sequence of positive numbers $\{M_n\}$ satisfying

For all $n \geq 1$, and all $x \in$:

$|f_n(x)| \leq M_n$ where $\sum_{n=1}^{\infty} M_n < \infty$.

Then the series

$\sum_{n=1}^{\infty} f_n(x)$ converges

absolutely and uniformly on

A.

For a series $\sum_{n=1}^{\infty} a_n$ to converge

absolutely means that

$\sum_{n=1}^{\infty} |a_n|$ converges

For example, $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$

converges, because $\sum_{n=1}^{\infty} \frac{1}{n^2}$

also converges.

For a series of functions $\{f_n\}$ on A to converge uniformly means that for any $\varepsilon > 0$, there is a constant N, so that $|S_m(x) - S_n(x)| < \varepsilon$, if both $m, n \geq N$.

Back to our differential equation:

Since $|y_n(x) - y_{n-1}(x)|$

$$\leq \frac{MK^{n-1} h^n}{n!},$$

it follows that the series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] \qquad (1)$$

converges absolutely and uniformly on the interval

$|x - x_0| \leq h$. Note that the k-th partial sum of the series

$$\sum_{n=1}^{k} [y_n(x) - y_{n-1}(x)]$$

$$= [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)]$$

$$\cdots + [y_k(x) - y_{k-1}(x)],$$

we see that

$$\sum_{n=1}^{k} \left[y_n(x) - y_{n-1}(x) \right] = y_k(x).$$

Thus the statement that the series (1) converges absolutely and uniformly is equivalent to the statement that the sequence $\{y_n(x)\}$ converges uniformly on the interval $|x-x_0| \le h$.

If we define $\phi(x) = \lim_{n\to\infty} y_n(x)$

and recall from the definition

of the sequence $y_n(x)$ that

each $y_n(x)$ is continuous

on $|x-x_0| \le h$, it follows

that (since the convergence

is uniform) that

$\phi(x)$ is also continuous

and

$$\phi(x) = \lim_{n\to\infty} y_n(x)$$

$$= y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt.$$

It follows that y_{n-1} converges

uniformly to $\phi(x)$, and hence

$f(t, y_{n-1}(t))$ converges

uniformly to $f(t, \phi(t))$.

Since definite integrals preserve uniform , we conclude that

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt,$$

which implies that $\phi(x)$ is a solution of the differential equation