Exam 1 Math 341 **Name** ______________________________

Spring 2018

1. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

Show true for $n = 1$.

$\frac{1}{6}(1)(2)(3) = \frac{1}{6}(6) = 1$, and $1^2 = 1$. Thus the identity is true for $n = 1$

Assume $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for a given $n \in \mathbb{N}$. Hence,

$1^2 + \cdots + n^2 + (n+1)^2$

$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$= \frac{n+1}{6}(n(2n+1) + 6(n+1))$

$= \frac{n+1}{6}(2n^2 + n + 6n + 6)$

$= \frac{n+1}{6}(2n^2 + 7n + 6)$

$= \frac{n+1}{6}(2n+3)(n+2)$

$= \frac{(n+1)(n+2)(2n+3)}{6}$

which proves the identity for $n + 1$

Thus the identity is true for all $n \in \mathbb{N}$.

2. (a) A sequence (x_n) is Cauchy if

for every $\epsilon > 0$ there is an integer $H \in \mathbb{N}$ such that if $m, n \geq H$ then $|x_m - x_n| < \epsilon$.

(b) Show that if (x_n) is convergent, then (x_n) is Cauchy.

Suppose that $\lim x_n = x$. Then if $\epsilon > 0$, there is an integer $K \in \mathbb{N}$ such that if $m \geq K$, then $|x_m - x| < \frac{\epsilon}{2}$. Similarly, $|x_n - x| < \frac{\epsilon}{2}$ if $n \geq K$. Hence,

$$|x_m - x_n| = |(x_m - x) - (x_n - x)| \leq |x_m - x| - |x_n - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

3. (a) A sequence of closed intervals $I_1, I_2, \cdots$ is nested if

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

(b) State the Nested Interval Property.

The Nested Interval Property states that there is a number $\eta \in \mathbb{R}$ such that $\eta \in I_n$ for all $n = 1, 2, \cdots$

(c) State the Bolzano-Weierstrass Theorem.

The Bolzano-Weierstrass Theorem states that if (x_n) is a closed bounded sequence, then there is a subsequence (x_{n_k}) that converges to a number x.

4. Let $x_1 = 6$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$.

(a) Show that (x_n) is bounded and decreasing.

It is clear that if $x_n > 0$, then $x_{n+1} > 0$. Thus (x_n) is bounded below.

Note also that $x_2 = 5$. To show that (x_n) is decreasing, note that $x_2 < x_1$. Suppose by induction that $x_{n+1} < x_n$.

Then $\frac{1}{2}x_{n+1} < \frac{1}{2}x_n$, so $\frac{1}{2}x_{n+1} + 2 < \frac{1}{2}x_n + 2$. Hence, $x_{n+2} < x_{n+1}$.

(b) Find the limit.

The Monetone Convergence Theorem implies that there is an x so that $\lim(x_n) = x$, which implies that $\lim(x_{n+1}) = x$. Hence, $x = \frac{1}{2}x + 2$ which implies that $x = 4$.

5. State and prove the Uniqueness of Limits.

The Uniqueness Theorem states that a sequence (x_n) can have at most one limit.

Suppose that a sequence has two distinct limits x' and x''. Let $\epsilon > 0$. Then there is an integer K_1 so that $|x_n - x'| < \frac{\epsilon}{2}$ if $n > K_1$ and an integer K_2 so that $|x_n - x''| < \frac{\epsilon}{2}$ if $n > K_2$.

If $n \geq K = \text{Max}\ \{K_1, K_2\}$ then

$|x' - x''| = |(x' - x_n) + (x'' - x_n)| \leq |x' - x_n| + |x'' - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$

Thus $|x' - x''| < \epsilon$ for all ϵ, which implies that $x' - x'' = 0$.

6. Show that if (x_n) is a bounded increasing sequence, then there is a number x such that $\lim(x_n) = x$.

Since (x_n) is a bounded increasing sequence, we can set $x =$sup S, where $S = \{x_n, n = 1, 2, \cdots\}$. Let $\epsilon > 0$. Then $x - \epsilon$ is not an upper bound. Hence, there is an integer k so that $x_k > x - \epsilon$. Since (x_n) is increasing, it follows that if $n \geq k$, then $x - \epsilon < x_k \leq x_n \leq x < x + \epsilon$. Hence we get that $x - \epsilon < x_n < x + \epsilon$ for all $n \geq k$.

Thus $\lim_{n\to\infty} x_n = x$.

7. State and prove the Product Rule for Sequences.

The Product Rule states that if (x_n) and (y_n) are sequences with $\lim(x_n) = x$ and $\lim(y_n) = y$ then $\lim(x_n y_n) = xy$.

Note that

$|x_n y_n - xy| = |(x_n - x)y_n + x(y_n - y)| = |(x_n y_n - xy_n) + (xy_n - xy)|$

$\leq |x_n - x||y_n| + |x||y_n - y|$

Let $M_1 = \sup \{|y_n|\}$ and $M_2 = |x|$. Let $M = \sup\{M_1, M_2\}$. (The Boundedness Theorem states that (y_n) is bounded.)

Choose K sufficiently large so that if $n \geq K$ then $|x_n - x| < \frac{\epsilon}{2M}$ and $|y_n - y| < \frac{\epsilon}{2M}$.

Hence $|x_n y_n - xy| < \frac{\epsilon}{2M}M + \frac{\epsilon}{2M}M < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Therefore, $\lim x_n y_n = xy$

8. Consider the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2+(-1)^n}$.

Does the series converge or diverge? To receive credit, you must state the theorems you use to justify your answer.

Let $a_n = \frac{\sqrt{1+\frac{1}{n^2}}}{1+\frac{(-1)^n}{n}}$. The original series can be written as $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

Using the Product Rule and the Sum Rule, it follows that $\lim(1+\frac{1}{n^2}) = 1$.

Since $\sqrt{x}$ is continuous at $x = 1$, we have $\lim \sqrt{1+\frac{1}{n^2}} = 1$.

Similarly the Squeeze Rule and the Sum Rule imply that $\lim(1+\frac{(-1)^n}{n}) = 1$, and then the Quotient Rule implies that $\lim \frac{\sqrt{1+\frac{1}{n^2}}}{1+\frac{(-1)^n}{n}} = \lim a_n = 1$

Recall that the series can be written as $\sum_{n=1}^{\infty} \frac{a_n}{n}$.

Since $\lim \frac{\frac{a_n}{n}}{\frac{1}{n}} = 1$ and since $\sum \frac{1}{n}$ diverges, it follows from the Limit Comparison Test that $\sum \frac{a_n}{n}$ diverges.

Therefore $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^2+(-1)^n}$ diverges.