MA341

An Introduction to Real Analysis

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The textbook is

An Introduction to Real Analysis
4th Edition
by Bartle and Sherbert

The course homepage is

math.purdue.edu/~catlin

Click on MA341 Web Page
under Spring Semester 2018

All lectures and homework
assignments will be posted online

In this course we will give a rigorous and detailed study of the ideas and techniques of calculus of one variable, including

1. Set Theory
2. Real Numbers
3. Sequences and Series
4. Limits
5. Continuous Functions

6. Differentiation

7. Integration

8. Sequences and Series of Functions

9. Taylor Series

1.1 Sets and Functions

If x is in a set A, we write

$x \in A$

We also say x is a member of A or that x belongs to A. If x is not in A, we write $x \notin A$.

If every element of a set A belongs to a set B, we say A is a subset of B, and

$A \subseteq B$ or $B \supseteq A$.

Some common sets of numbers are:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ integers

$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z},\ n \neq 0\}$

rational numbers

$\mathbb{R}$: set of real numbers

Sometimes a set A is obtained by specifying a property that determines the elements of A.

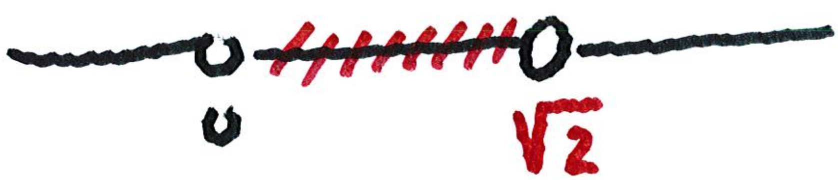
Ex. We say n is an even integer if there is an integer k, so that $n = 2k$.

$$E = \{ n \in \mathbb{Z} : n = 2k, \text{ for any } k \in \mathbb{Z} \}$$

or

$$E = \{ 2k : k \in \mathbb{Z} \}$$

Ex. Let $I = \left\{ x \in Q : \begin{array}{c} 0 < x \\ \text{and} \\ x^2 < 2 \end{array} \right\}$

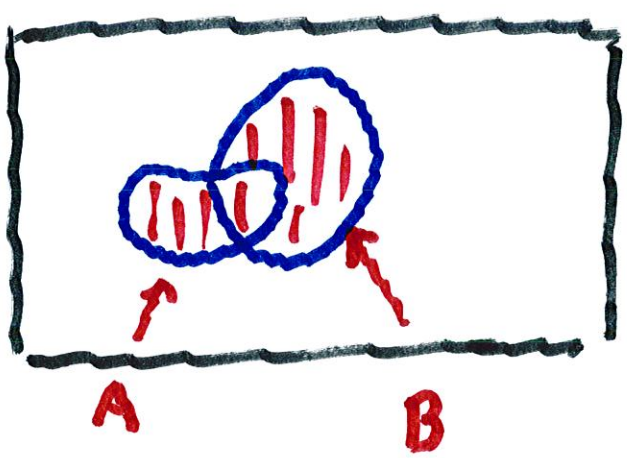


Set Operations

Def (a). The <u>union</u> of sets A and B is

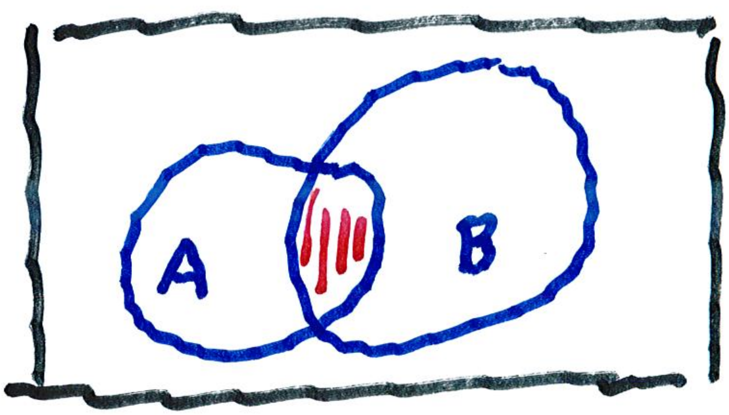
$$A \cup B = \{ x : x \in A \text{ or } x \in B \}$$

(x can be in both)



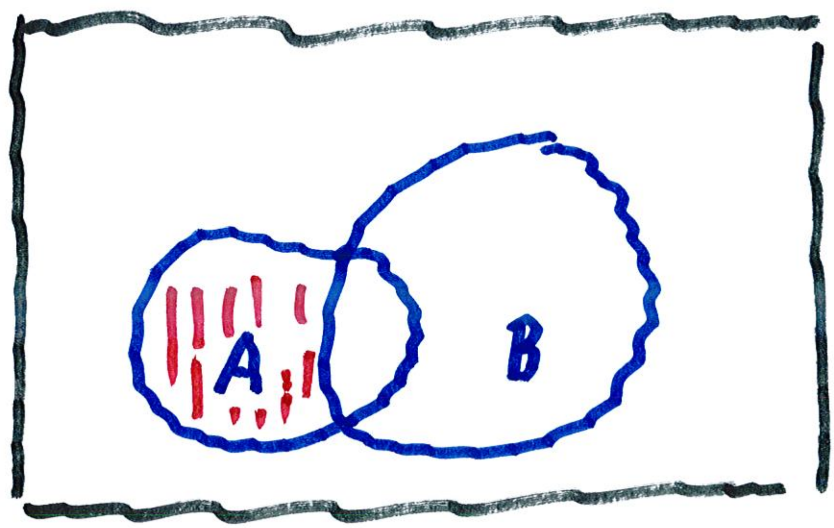
(b) The intersection of the sets A and B is the set

$$A \cap B = \{x ; x \in A \text{ and } x \in B\}$$



(c) The complement of B relative to A is the set

$$A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$$



The set with no elements is the empty set, written as $\emptyset$

Two sets A and B are said to be disjoint if there is no element in both A and B.

A and B are disjoint if $A \cap B = \emptyset$

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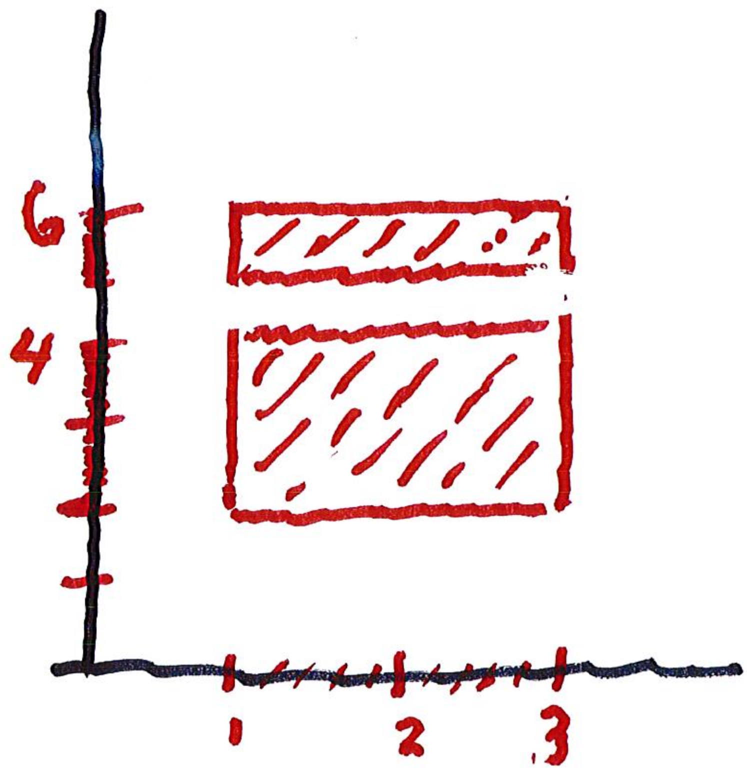
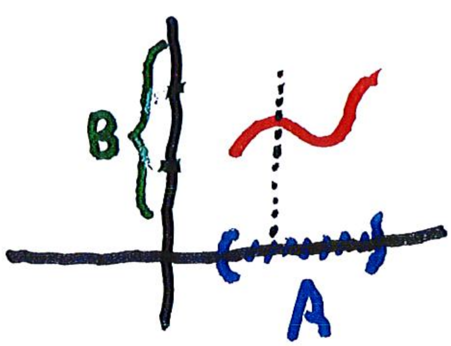
$$A \times B = \{(a, b) : a \in A,\ b \in B\}$$

If $A = \{x : 1 \leq x \leq 3\}$

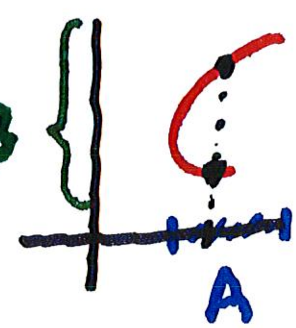
and $B = \{y : 2 \leq y \leq 4 \text{ or } 5 \leq y \leq 6\}$



A function f from A to B is a set f of ordered pairs in $A \times B$ such that for each a in A, there is unique b in B such that $(a,b) \in f$

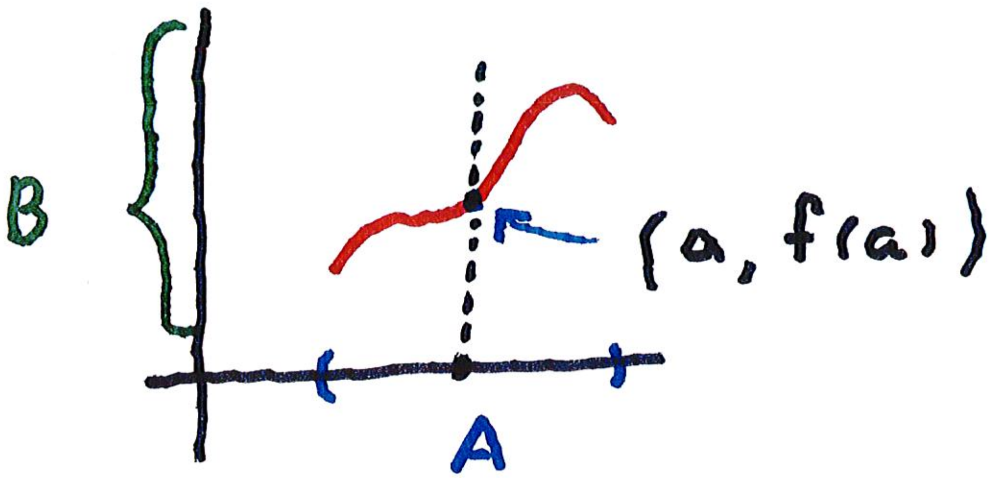


is a fcn.



not a fcn.

If $(a, b) \in f$, we often write $f(a) = b$



We write Domain $= D(f) = A$

Also $R(f) = \left\{ f(a) : a \in A \right\}$

Composition of Fcns.

If A, B, and C are maps,

and $f: A \rightarrow B$ and $g: B \rightarrow C$

then the composition of f and g is

$$(g \circ f)(x) = g(f(x))$$

for all x in A

Ex. Suppose $f(x) = x^4 - 1$
for x in
$(-\infty, \infty)$

and $g(x) = \sqrt{x}$, for

$0 \le x < \infty$,

then we cannot form

$$(g \circ f)(x) = \sqrt{x^4 - 1}.$$

The problem is $x^4 - 1 < 0$

if $-1 < x < 1$,

because $\sqrt{x^4-1}$ only makes sense

if $x^4 - 1 \ge 0$, i.e., if $|x| \ge 1$.

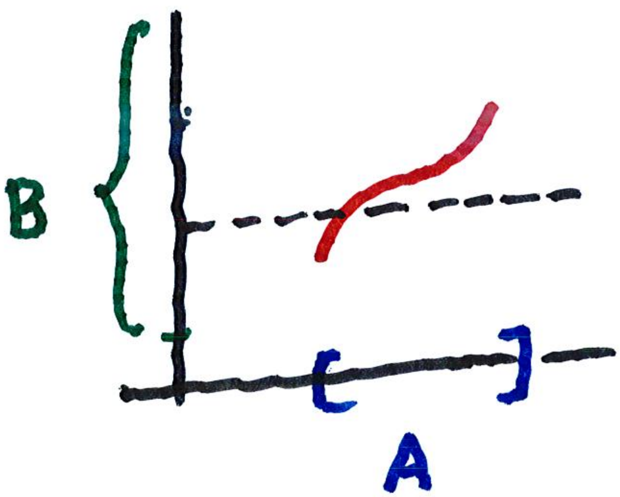
Then we modify f by defining $f(x) = x^4 - 1$ for $|x| \geq 1$.

Definition. A function $f: A \to B$ is injective, if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. (f is 1-to-1)

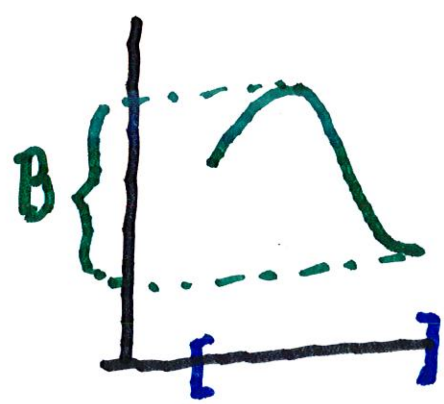
Equivalently, if whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Also $f: A \to B$ is surjective if whenever $y \in B$, then there is an x in A so $f(x) = y$

(f is onto)



f is 1-to-1 but not onto



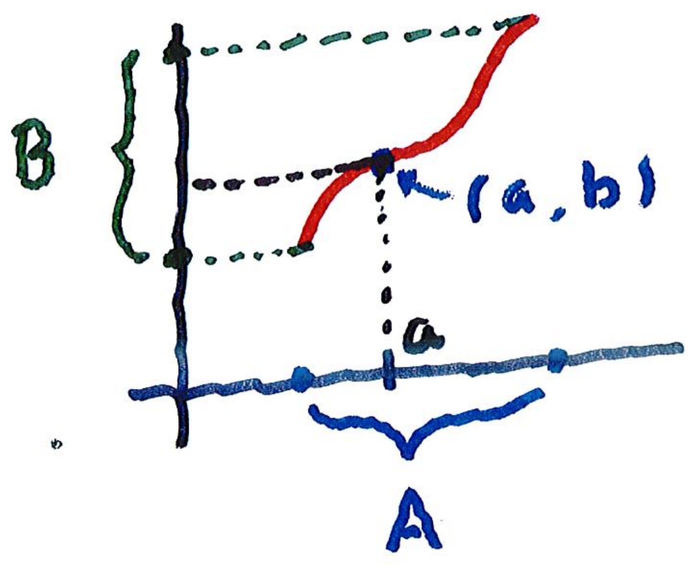
f is onto but not 1-to-1

Lecture 1 cont'd:

We say f is bijective

if f is both injective

and surjective.



$f(a) = b$

$g(b) = a.$

Theorem. Suppose $f: A \longrightarrow B$

is bijective, (i.e., both
onto and
1-to-1)

Then there is a bijection

$g: B \longrightarrow A$ that satisfies

(a) $g(f(a)) = a$ for all
a in A

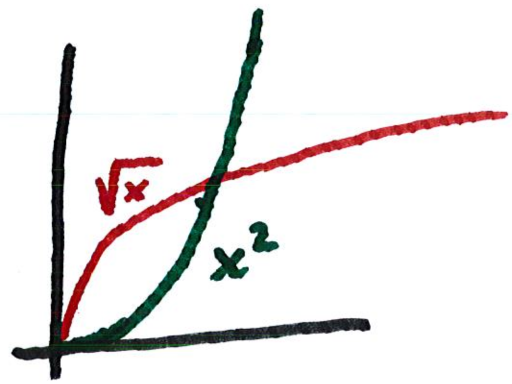
(b) $f(g(b)) = b$ for all
b in B

We write $g = f^{-1}$ and $f = g^{-1}$

The formula in (a) shows that g is surjective. For any a in A, $f(a)$ is the value of x such that $g(x) = a$.

(b) shows that g is injective. For if $g(b_1) = g(b_2)$. Then $f(g(b_1)) = f(g(b_2))$, so that $b_1 = b_2$.

Ex. Let $f(x) = x^2$ $(0 \le x < \infty)$. The inverse of f is $\sqrt{x}$.



$x^2 = 3$

Apply $\sqrt{}$. $\sqrt{x^2} = \sqrt{3}$

or $x = \sqrt{3}$.

Ex. $\sin x$ maps $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1,1]$

$\sin^{-1}$ maps $[-1,1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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Suppose $\sin x = .42$

Apply $\sin^{-1}$:

$$\sin^{-1}(\sin x) = \sin^{-1}(.42)$$

$$\rightarrow \quad x = \sin^{-1}(.42)$$

Ex. Let $A = \{x \in \mathbb{R} : x \neq -1\}$

and let $f(x) = \dfrac{2x-1}{x+1}$.

Show that f is injective.

Suppose $f(x_1) = f(x_2)$

$$\frac{2x_1 - 1}{x_1 + 1} = \frac{2x_2 - 1}{x_2 + 1}$$

$$(2x_1 - 1)(x_2 + 1) = (2x_2 - 1)(x_1 + 1)$$

$$2x_1 - x_2 = -x_1 + 2x_2$$

$$\rightarrow \quad 3x_1 = 3x_2$$

or $x_1 = x_2$. ✓

Now find the range of f.

Find all y, such that

$$y = \frac{2x-1}{x+1} \to yx + y = 2x - 1$$

Solve for x: $(y-2)x = -y-1$

$$\to \quad x = \frac{y+1}{2-y}$$

This can be solved only if $y \neq 2$. $R(f) = \{y \in \mathbb{R} : y \neq 2\}$