3.2 Limit Thms.

Given 2 sequences

$X = (x_n)$ and $Y = (y_n)$ such that

$\lim (x_n) = x$ and $\lim (y_n) = y$,

we proved that

1. $\lim (x_n + y_n) = x + y$

2. $\lim (x_n y_n) = xy$.

3 To prove $\lim (c x_n) = cx$,

let $Y = (y_n) = c$, for all c.

Then $\lim cx_n = \lim y_n x^n$

$= \lim y_n \cdot \lim x_n$

$= cx$ $\therefore \lim cx_n = cx.$

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4. Now suppose $Z = (z_n)$ and that $\lim (z_n) = z \neq 0$.

Choose $K_1 \in \mathbb{N}$ so that if $n \geq K_1$, then $|z_n - z| < \frac{|z|}{2}$.

It follows that

$$|z_n| = |(z_n - z) + z|$$

$$= |z + (z_n - z)|$$

$$\geq |z| - |z_n - z|$$

$$\geq |z| - \frac{|z|}{2} = \frac{|z|}{2}.$$

We use this to estimate the limit of $\frac{1}{z}$:

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| = \left| \frac{z - z_n}{z_n z} \right|$$

$$\leq |z - z_n| \cdot \frac{2}{|z|^2}$$

since $\frac{1}{|z_n|} \leq \frac{2}{|z|}$ when

$n \geq K_1$. Now choose $\varepsilon > 0$

and choose K_2 so that

$|z_n - z| < \frac{|z|^2}{2} \varepsilon$ when $n \geq K_2$.

Now set $K = \mathrm{Max}\{K_1, K_2\}$.

If $n \geq K$, then

$$\left| \frac{1}{z_n} - \frac{1}{z} \right| \le |z - z_n| \cdot \frac{2}{|z|^2}$$

$$< \frac{|z|^2}{2} \cdot \varepsilon \cdot \frac{2}{|z|^2} = \varepsilon$$

This shows that $\lim \left(\frac{1}{z_n} \right) = \frac{1}{z}$.

Ex. Use the Limit Laws to compute $\lim \dfrac{n^2 + 2n}{3n^2 + 1}$

$$\frac{n^2 + 2n}{3n^2 + 1} = \frac{n^2 \left(1 + \frac{2}{n}\right)}{n^2 \left(3 + \frac{1}{n^2}\right)}$$

$$= \frac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}}.$$

Since $\lim \frac{1}{n} = 0$,

we have $\lim \frac{2}{n} = 0$ and $\lim \frac{1}{n^2} = 0$

$\therefore \lim \left(1 + \frac{2}{n}\right) = 1 + 0 = 1$

and $\lim \left(3 + \frac{1}{n^2}\right) = 3$

Hence the Quotient Rule

$$\rightarrow \lim \frac{1 + \frac{2}{n}}{3 + \frac{1}{n^2}} = \frac{1}{3}$$

To show that $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

we first show: $\sqrt{}$ is increasing:

If $0 < a < b$, then $0 < \sqrt{a} < \sqrt{b}$

Suppose that $\sqrt{a} \geq \sqrt{b}$.

Then $a = \sqrt{a}\sqrt{a} \geq \sqrt{b}\sqrt{a} \geq \sqrt{b}\sqrt{b} = b$.

This contradicts the hypothesis that $a < b$.

We now can prove:

$$\lim \frac{1}{\sqrt{n}} = 0$$

Proof: Choose $\varepsilon > 0$. Then choose an integer K so that $K > \frac{1}{\varepsilon^2}$. If $n \in \mathbb{N}$ and $n \geq K$, then $n \geq K > \frac{1}{\varepsilon^2}$. This gives $\sqrt{n} > \sqrt{\frac{1}{\varepsilon^2}} = \frac{1}{\varepsilon}$, which gives $\frac{1}{\sqrt{n}} < \varepsilon$, which implies $\left| \frac{1}{\sqrt{n}} - 0 \right|$ if $n \geq K$. We conclude that $\lim \frac{1}{\sqrt{n}} = 0$.

Ex. Compute $\lim \frac{\sqrt{n}}{2n+3}$

Factor out highest power

$$= \frac{\sqrt{n} \cdot 1}{n\left(2+\frac{3}{n}\right)} = \frac{1}{\sqrt{n}} \cdot \frac{1}{\left(2+\frac{3}{n}\right)}$$

Note $\lim \frac{1}{\sqrt{n}} = 0$ and

$\lim \frac{1}{2+\frac{3}{n}} = \frac{1}{2}$.

$\therefore$ Product Rule implies

$$\lim \frac{\sqrt{n}}{2n+3} = 0 \cdot \frac{1}{2} = 0$$

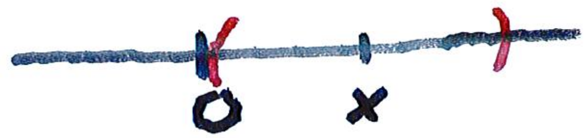
Thm. Suppose $\lim x_n = x$

and that $x_n \leq 0$. Then

$x \leq 0$.

Pf. Suppose statement is

not true, i.e., suppose $x > 0$.



Pick $\varepsilon = x$

Then there is K, so if

$n \geq K$, then $|x_n - x| < x$

Hence $-\varepsilon < x_n - x < \varepsilon$.

$-x < x_n - x \to 0 < x_n$.

This contradicts hypothesis that $x_n \leq 0$

Corollary. Suppose (x_n) and (y_n) are both convergent and that $x_n \leq y_n$, all n. Then $x \leq y$

Pf. Set $z_n = x_n - y_n$.

Then $z_n \le 0$, for all n.

Hence the theorem implies

$\lim z_n = z$, i.e., $z \le 0$.

$\therefore\ x - y \le 0$.

i.e. $\lim(x_n) \le \lim(y_n)$.

Suppose $a \le x_n \le b$ and

that (x_n) is convergent.

Then $a \le \lim(x_n) \le b$.

Pf. To prove $\lim(x_n) \le b$,

set $(y_n) = (b)$ for all n. To prove

$\lim(x_n) \ge a$, set

The hypothesis that $x_n \le b$

(using previous result)

implies that $\lim(x_n) \le \lim(y_n)$,

or $\lim(x_n) \le b$.

Similarly, if we set $y_n = a$,

for all n, then the corollary

[illegible] $\Rightarrow$ $y_n \le x_n$, which implies $a \leq \lim(X_n)$.

We now prove:

Squeeze Thm.

Suppose that $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ are sequences with

$$x_n \leq y_n \leq z_n$$

Suppose also that $\lim(x_n) = \lim(z_n)$.

Then $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

Proof: Let $w = \lim(x_n) = \lim(z_n)$.

For any $\varepsilon > 0$, choose K so

that if $n \geq K$, then

$|x_n - w| < \varepsilon$ and $|z_n - w| < \varepsilon$.

$$-\varepsilon < x_n - w \leq y_n - w \leq z_n - w < \varepsilon$$

$$\rightarrow -\varepsilon < y_n - w < \varepsilon .$$

Since ε is arbitrary, it follows

that $\lim(y_n) = w$.

Ex. Show $\lim \left((-1)^n \cdot \frac{1}{2n+1} \right) = 0$

Show $x_n = -\frac{1}{2n+1}$ and

$y_n = \frac{(-1)^n}{2n+1}$.

Then $z_n = \frac{1}{2n+1}$.

Then $x_n \leq y_n = z_n$, all n.

and $\lim x_n = 0$, $\lim z_n = 0$.

$\therefore$ Squeeze Thm. $\Rightarrow \lim \frac{(-1)^n}{2n+1}$

converges.

Ratio Test for Sequences.

Thm. Suppose that (x_n) is a sequence of positive numbers such that $L = \lim\left(x_{n+1}/x_n\right)$ exists. If $L < 1$, then

$\lim(x_n) = 0$.

Note that $L \geq 0$. Let ε be chosen so that

$0 < L-\varepsilon < L < L+\varepsilon$, where

$L+\varepsilon \leq R < 1$.