3.3 Monotone Sequences

Definition. We say a sequence (x_n) is increasing if

$$x_n \le x_{n+1}, \qquad \text{all } n = 1, 2, \ldots$$

That is

$$x_1 \le x_2 \le \cdots x_n \le x_{n+1} \le \cdots$$

We say (y_n) is decreasing if

$$y_n \ge y_{n+1}, \qquad n = 1, 2, \ldots$$

That is

$$y_1 \ge y_2 \ge \cdots \ge y_n \ge y_{n+1} \ge \cdots$$

If (x_n) is increasing or decreasing, we say (x_n) is monotone.

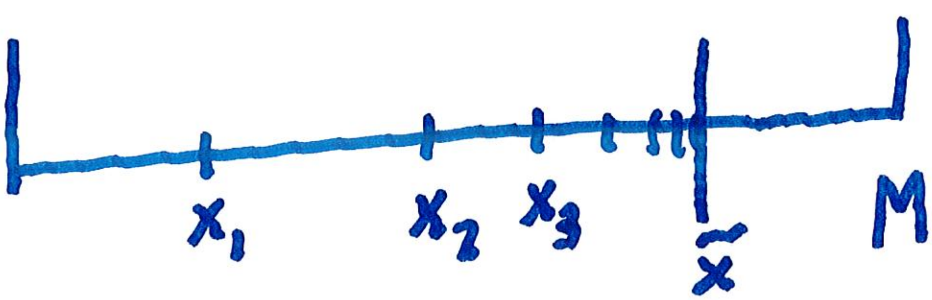
Monotone Convergence Thm.

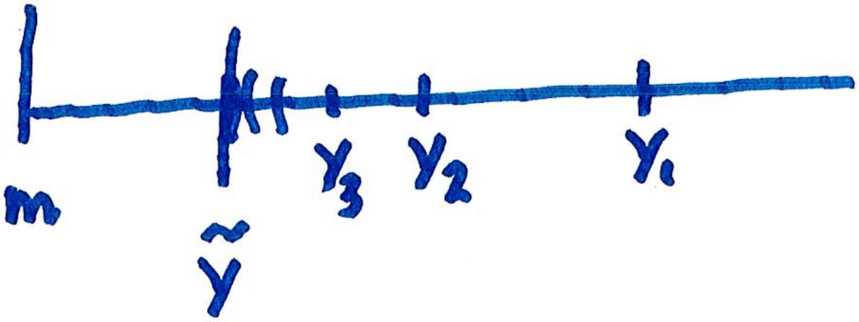
If (x_n) is a bounded monotone sequence, then it converges. In fact, if (x_n) is increasing and bounded, then

$$\lim(x_n) = \tilde{x} = \sup\{x_n : n \in N\}$$

Also, if (y_n) is decreasing and bounded, then

$$\lim(y_n) = \tilde{y} = \inf\{y_n : n \in N\}$$

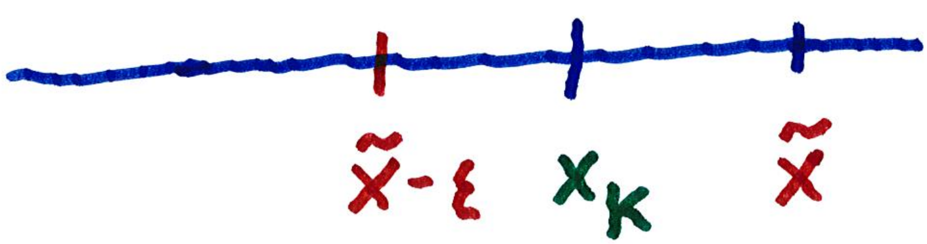




Proof. Since $x_n \le M$ for all $n \in N$, we define

$$\tilde{x} = \sup\{x_n : n \in N\}.$$

For any $\varepsilon > 0$, $\tilde{x} - \varepsilon$ is not an upper bound. It follows that there is a $K \in N$, such that $\tilde{x} - \varepsilon < x_K \le \tilde{x}$.



Since (x_n) is increasing,

if $n \geq K$, then

$$\tilde{x} - \varepsilon < x_K \leq x_n \leq \tilde{x} < \tilde{x} + \varepsilon,$$

where the inequality

$x_n \leq \tilde{x}$ comes from the fact

that $\tilde{x}$ is an upper bound

of $\left\{ x_n : n \in \mathbb{N} \right\}$

It follows that $|x_n - \tilde{x}| < \varepsilon$

if $n \geq K$. Hence $\lim(x_n) = \tilde{x}$.

In the case of (y_n),

$y_n \geq m$ for all n, which

implies that there is a

number $\tilde{y} = \inf\{y_n : n \in N\}$

For any $\varepsilon > 0$, there is a K'

so that $\tilde{y} \leq y_K < \tilde{y} + \varepsilon$.

Since (y_n) is decreasing,

we obtain that if $n \geq K'$, then

$$\tilde{y} + \varepsilon > y_{K'} \geq y_n \geq \tilde{y} > \tilde{y} - \varepsilon,$$

or that

$$\tilde{y} - \varepsilon < y_n < \tilde{y} + \varepsilon, \text{ for } n \geq K'$$

It follows that $\lim(y_n) = \tilde{y}$,

which proves the theorem.

We now use the Least Upper Bound Property to evaluate the lim of some sequences.

Ex. Let $0 < r < 1$. Then

$$\lim r^n = 0.$$

Note that (r^n) is decreasing. Since $r^n > 0$, it follows

that $\underline{\lim r^n = R}$, where $R \geq 0$.

In fact, for any ϵ, there is a K, so that if $n \geq K$, then

$|r^n - R| < \varepsilon$.

Since $n+1 \geq K$, it follows that $r^{n+1} \geq K$. Hence,

$|r^{n+1} - R| < \varepsilon$, which implies that $\lim r^{n+1} = R$.

On the other hand,

$$\lim (r^{n+1}) = \lim (r^n \cdot r)$$
$$= R \cdot r$$

Since $R = Rr$, it follows

that $R = 0$. Thus, we have

proved that $\lim r^n = 0$.

Ex. Define $y_{n+1} = \frac{2}{5} y_n + 1$,

with $y_1 = 1$.

Assume first

that $\lim y_n = y$. Then we have

$$y = \frac{2}{5} y + 1 \rightarrow \frac{3}{5} y = 1$$

$$\rightarrow \underline{y = \frac{5}{3}}.$$

Use induction to show that

if $1 \le y_n \le 4$, then y_{n+1} also satisfies $1 \le y_{n+1} \le 4$.

In fact, if $y_n \ge 1$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \ge \frac{2}{5} \cdot 1 + 1 > 1$$

Similarly, if $y_n \le 4$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \le \frac{8}{5} + 1 \le \frac{13}{5}.$$

Now we show that $\underline{y_{n+1} > y_n}$

This is obvious when $n = 1$.

Now assume that $y_{n+1} > y_n$.

Then $\frac{2}{5} y_{n+1} > \frac{2}{5} y_n$,

which gives

$$\frac{2}{5} y_{n+1} + 1 > \frac{2}{5} y_n + 1$$

or $\underline{y_{n+2} > y_{n+1}}$.

Since $y_n \leq 4$ for all $n \in \mathbb{N}$,

and since (y_n) is increasing,

we conclude that there

is a $y \in [1, 4]$ such that

$\lim (y_n) = y$ and $\lim (y_{n+1}) = y$.

This implies that

$$y = \frac{2}{5} y + 1 \rightarrow y = \frac{5}{3}.$$

Ex. Study the convergence of

$$y_n = \left\{ \frac{1}{n+1} + \cdots + \frac{1}{2n} \right\}.$$

Note that

$$y_{n+1} = -\frac{1}{n+1} + \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \right\}$$

$$\rightarrow y_{n+1} = y_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= y_n + \frac{1}{(2n+1)(2n+2)}$$

$$\therefore \; y_{n+1} > y_n$$

Note also that

$$y_n \le n \cdot \frac{1}{n+1} \le 1 \text{ for all } n.$$

Hence the Monotone Convergence Theorem implies that

$$y_n \to y \le 1 \text{ as } n \to \infty.$$

HW # 1, p. 77.

Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2} x_n + 2$.

Show that (x_n) is bounded

and decreasing. Find the limit.

$$x_{n+1} = \frac{1}{2} x_n + 2$$

$$x_n = \frac{1}{2} x_{n-1} + 2.$$ Subtracting:

$$x_{n+1} - x_n = \frac{1}{2} (x_n - x_{n-1}). \quad (1)$$

$\therefore$ Monotone Conv. Thm implies

$(x_n) \to s$, some $s > 0$.

$\therefore$ $s = \frac{1}{2}s + 2. \Rightarrow s = 4$

HW #2. Let $x_1 > 1$ and

$x_{n+1} = 2 - \frac{1}{x_n}$. Show (x_n) is

monotone and bounded.

Clearly $x_{n+1} = 2 - \frac{1}{x_n}$, and

$$x_n = 2 - \frac{1}{x_{n-1}}.$$

Subtracting, we get

$$x_{n+1} - x_n = -\frac{1}{x_n} + \frac{1}{x_{n-1}}$$

$$= \frac{x_n - x_{n-1}}{x_n x_{n-1}}$$

Since $x_1 > 1$, we have by induction that $x_n > 1$, for all n.