Another Problem using the Monotone Sequence Theorem

Ex. # 2., page 77.

Let $x_1 > 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$

Find $\lim (x_n)$.

First, note that if $x_n > 1$,

then $\frac{1}{x_n} < 1$, so that

$x_{n+1} = 2 - \frac{1}{x_n} > 1$. Hence

$x_n > 1$ for all $n = 1, 2, \dots$.

We want to show that (x_n) is decreasing. We have

$$x_1 - x_2 = x_1 - \left(2 - \frac{1}{x_1}\right) = \frac{(x_1 - 1)^2}{x_1} > 0.$$

Similarly, we have :

$$x_{n+1} - x_{n+2} = \left(2 - \frac{1}{x_n}\right) - \left(2 - \frac{1}{x_{n+1}}\right)$$

$$= \left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{x_n - x_{n+1}}{x_n x_{n+1}}$$

$$< (x_n - x_{n+1})$$

where the final inequality follows from $x_n > 1$, $x_{n+1} > 1$ for all n. Since (x_n) is decreasing it follows from the Monotone Convergence Theorem that $\tilde{x} = \lim(x_n)$ exists, which implies that $\lim(x_n) \geq 1$.

We conclude that $\tilde{x} = 2 - \frac{1}{\tilde{x}}$, which yields that

$$(\tilde{x}-1)^2 = 0, \quad \text{i.e.,} \quad \tilde{x} = 1.$$

3.4. Subsequences

Let $X = (x_n)$ be a sequence

and let

$$n_1 < n_2 < \dots < n_k < \dots$$

be a strictly increasing

sequence of integers in $\mathbb{N}$.

Then the sequence

$X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \dots)$$

is called a <u>subsequence</u>

of X.

Ex. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$

is a subsequence of

$\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) = X$

corresponding to $n_k = 2k$.

But $\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \ldots\right)$

is not a subsequence of X.

The following theorem is useful.

Thm. Suppose $X = (x_n)$ converges to x. If (x_{n_k}) is any subsequence of X, then

$$\lim_{k \to \infty} (x_{n_k}) = x.$$

Pf. Let $\varepsilon > 0$ and let $K(\varepsilon) > 0$ be such that if $n \geq K(\varepsilon)$, then $|x_n - x| < \varepsilon$.

Since

$n_1 < n_2, < \ldots < n_k < \ldots$

is an increasing sequence of natural numbers, it is easy to prove by induction that $n_k \geq k$.

Hence if $k \geq K(\varepsilon)$, then

$$n_k \geq k \geq K(\varepsilon),$$

so that $|x_{n_k} - x| < \varepsilon$.

Thus (x_{n_k}) also converges to x.

The following theorem is fundamental to the theory of calculus.

Bolzano-Weierstrass Thm.

A bounded sequence of real numbers has a convergent subsequence.

Pf. Since $\{x_n : n \in N\}$ is bounded, this set is contained in an interval $I_1 = [a_1, b_1]$

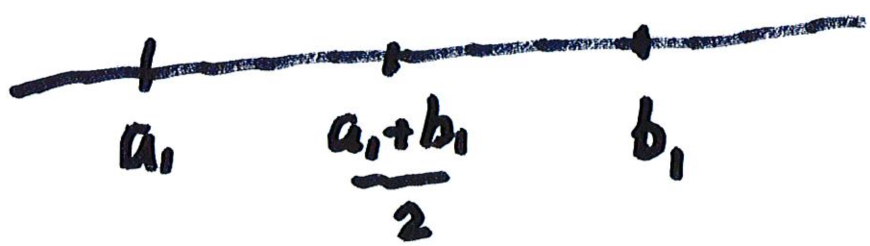
We set $n_1 = 1$.

We now bisect I_1 into two intervals I_1' and I_1''.

More precisely,

$$I_1' = \left[a_1 , \frac{a_1+b_1}{2}\right] \quad \text{and}$$

$$I_1'' = \left[\frac{a_1+b_1}{2} , b_1\right].$$



We divide $\{n \in N : n > n_1\}$

into two sets,

$$A_1 = \{n \in N : n > n_1 , x_n \in I_1'\}$$

$$B_1 = \{n \in N : n > n_1 , x_n \in I_1''\}$$

one of which is infinite.

In fact $A_1 \cup B_1$ contains every element of N except for n with $1 \leq n \leq n_1$.

According to our construction,

$$\left\{ n \in N : n > n_2,\ x_n \in I_2 \right\}$$

is infinite.

If A_1 is infinite, then we set $I_2 = I_1'$, and we let n_2 be the smallest natural number in A_1. Note that $x_{n_2} \in I_2$.

If A_1 is a finite set, then B_1 must be infinite, we let n_2 be the smallest natural number in B_1, and we set $I_2 = I_1''$.

We now bisect I_2 into subintervals I_2' and I_2''

and we divide the set

$$\{n \in \mathbb{N} : n > n_2,\ x_n \in I_2\}$$

into 2 parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2,\ x_n \in I_2'\}$$

$$B_2 = \{n \in \mathbb{N} : n > n_2,\ x_n \in I_2''\}$$

If A_2 is infinite, we

take $I_3 = I_2'$, and we let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 = I_2''$, and we let n_3 be the smallest natural number in B_2. Note that $x_{n_3} \in I_3$.

We continue in this way to obtain a sequence of nested intervals

$I_1 \supset I_2 \supset \dots \supset I_k \supset \dots$

and we obtain a subsequence $\{x_{n_k}\}$ of X such that

$x_{n_k} \in I_k$ for $k \in \mathbb{N}$.

In addition, for each k, 13.1

the set

$$\left\{ n \in \mathbb{N} : n > n_k ,\ x_n \in I_k \right\}$$

is infinite. This fact guarantees that when we split the interval I_k into I_k' and I_k'', one of the corresponding sets is nonempty.

By the Nested Interval Property, there is a point η such that

$$\eta \in \bigcap_{k=1}^{\infty} I_k .$$

The length of I_k is $\dfrac{(b-a)}{2^{k-1}}$. Since both x_{n_k} and η both lie in I_k, it follows that

$$|x_{n_k} - \eta| \le \frac{(b-a)}{2^{k-1}},$$

which implies that the subsequence $\{x_{n_k}\}$ of X converges to η.