3.5 Cauchy's Criterion.

Def'n. A sequence $X = (x_n)$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists a number H in $\mathbb{N}$ so that if $n, m \geq H$, then

$|x_n - x_m| < \varepsilon$

Even though the definition does not mention a limit x, still, the numbers x_n and x_m get closer as $n, m \to \infty$

Lemma. If a sequence approaches a limit x, then the sequence (x_n) is Cauchy

Proof of Lemma. If $x = \lim(x_n)$, then given $\varepsilon > 0$, there is a natural number K; such that if $n \geq K$, then $|x_n - x| < \frac{\varepsilon}{2}$.

Thus, if $n, m \geq K$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)|$$

$$\leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

it follows that (x_n) is a

Cauchy sequence.

Lemma. A Cauchy sequence is bounded.

Pf. Let $X = (x_n)$ be Cauchy, and set $\varepsilon = 1$. There is H in $\mathbb{N}$ so that, if $n \geq H$, then

$$|x_n - x_H| < 1.$$

By the Triangle Inequality, we have

$$|x_n| = |x_H + (x_n - x_H)|$$

$$\leq |x_H| + 1$$

If we set

$$M := \max\left\{ |x_1|, |x_2|, \dots |x_{H-1}|, |x_H| + 1 \right\},$$

then it follows that

$$|x_n| \le M, \quad \text{for all } n.$$

Cauchy Convergence Thm.

A sequence $X = (x_n)$ is convergent if it is a Cauchy sequence. We have to find x !!

We already showed that if X is convergent, then it is Cauchy. To prove the other direction, suppose X is Cauchy. We showed above that X is therefore bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence

$X' = (x_{n_k})$ of X that converges to a number x^*.

We will show that $\lim x_n = x^*$.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$, there is a natural number H such that if $n, m \geq H$, then $|x_n - x_m| < \frac{\varepsilon}{2}$. (1)

Since the subsequence

$X' = \{x_{n_k}\}$ converges to x^*,

there is a natural number

$K \geq H$ that belongs to the

set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \frac{\varepsilon}{2}$$

Since $K \geq H$, it follows

from (1) with $m = K$ that

$$|x_n - x_K| < \frac{\varepsilon}{2} \quad \text{for } n \geq H.$$

Therefore, if $n \geq H$,

we have

$$|x_n - x^*| = |(x_n - x_K) + (x_K - x^*)|$$

$$\leq |x_n - x_K| + |x_K - x^*|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\lim(x_n) = x^*$.

Ex. The polynomial equation

$x^3 - 5x + 1 = 0$ has a root

r with $0 < r < 1$.

We define an iteration

procedure to construct a

sequence (x_n) that

approaches the root r.

We define x_1 to be any

number with $0 < x_1 < 1$.

and we define

$$x_n^3 - 5x_{n+1} + 1 = 0,$$

or $x_{n+1} = \frac{1}{5}(x_n^3 + 1)$

It is easy to verify that if

$0 \le x_n \le 1$, then $0 \le x_{n+1} \le 1$.

Hence, $0 \le x_n \le 1$ for all $n \ge N$.

We can estimate $|x_{n+2} - x_{n+1}|$ by

$$|x_{n+2} - x_{n+1}| =$$

$$= \left| \frac{1}{5}(x_{n+1}^3 + 1) - \frac{1}{5}(x_n^3 + 1) \right|$$

$$= \frac{1}{5} \left| x_{n+1}^3 - x_n^3 \right|$$

$$= \frac{1}{5} \left| x_{n+1}^2 + x_{n+1} x_n + x_n^2 \right| \left| x_{n+1} - x_n \right|$$

$$\le \frac{3}{5} \left| x_{n+1} - x_n \right| .$$

Definition. We say that a sequence (x_n) of real numbers is <u>contractive</u> if there is a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \le C |x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. C is the constant of the sequence.

We now prove

Thm. Every contractive sequence is a Cauchy Sequence and therefore is convergent.

(Observe that in the above example, (x_n) is contractive with $C = \frac{3}{5}$.)

Pf. Using the contractive inequality, we get:

$$
\begin{aligned}
|x_{n+2} - x_{n+1}| &\le C|x_{n+1} - x_n| \\
&\le C^2 |x_n - x_{n-1}| \\
&\ \vdots \\
&\le C^n |x_2 - x_1|
\end{aligned}
$$

To show that (x_n) is Cauchy, we have

Hence

$$|x_m - x_n| \qquad (2)$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq \left(C^{m-2} + \dots + C^{n-1}\right)|x_2 - x_1|$$

$$= C^{n-1}\left(\frac{1 - C^{m-n}}{1-C}\right)|x_2 - x_1|$$

$$\leq C^{n-1}\left(\frac{1}{1-C}\right)|x_2 - x_1|$$

We conclude that (x_n)

is a Cauchy sequence

and therefore convergent.

Observe that we can estimate

the accuracy of (x_n) :

$$|x_m - x_n| \leq \frac{C^{n-1}}{1-C} |x_2 - x_1|$$

Since $\lim (x_m) = \tilde{x}$, we have

$$|\tilde{x} - x_n| \leq \frac{C^{n-1}}{1-C} |x_2 - x_1|$$

which shows that the error approaches 0 exponentially.

Since $x_{n+1} = \frac{1}{5}(x_n^3 + 1)$, we can take the limit as $n \to \infty$, so

$$x = \frac{1}{5}(x^3 + 1)$$

$\to$ $5x = x^3 + 1$, or

$$x^3 - 5x + 1 = 0$$