5.2 Combinations of Continuous Functions

Recall that a function

$f : A \to \mathbb{R}$ is continuous

at c if $\displaystyle\lim_{x \to c} f(x) = f(c)$.

If we define

$$V_\delta(c) = \left\{ x \in \mathbb{R} : |x - c| < \delta \right\}$$

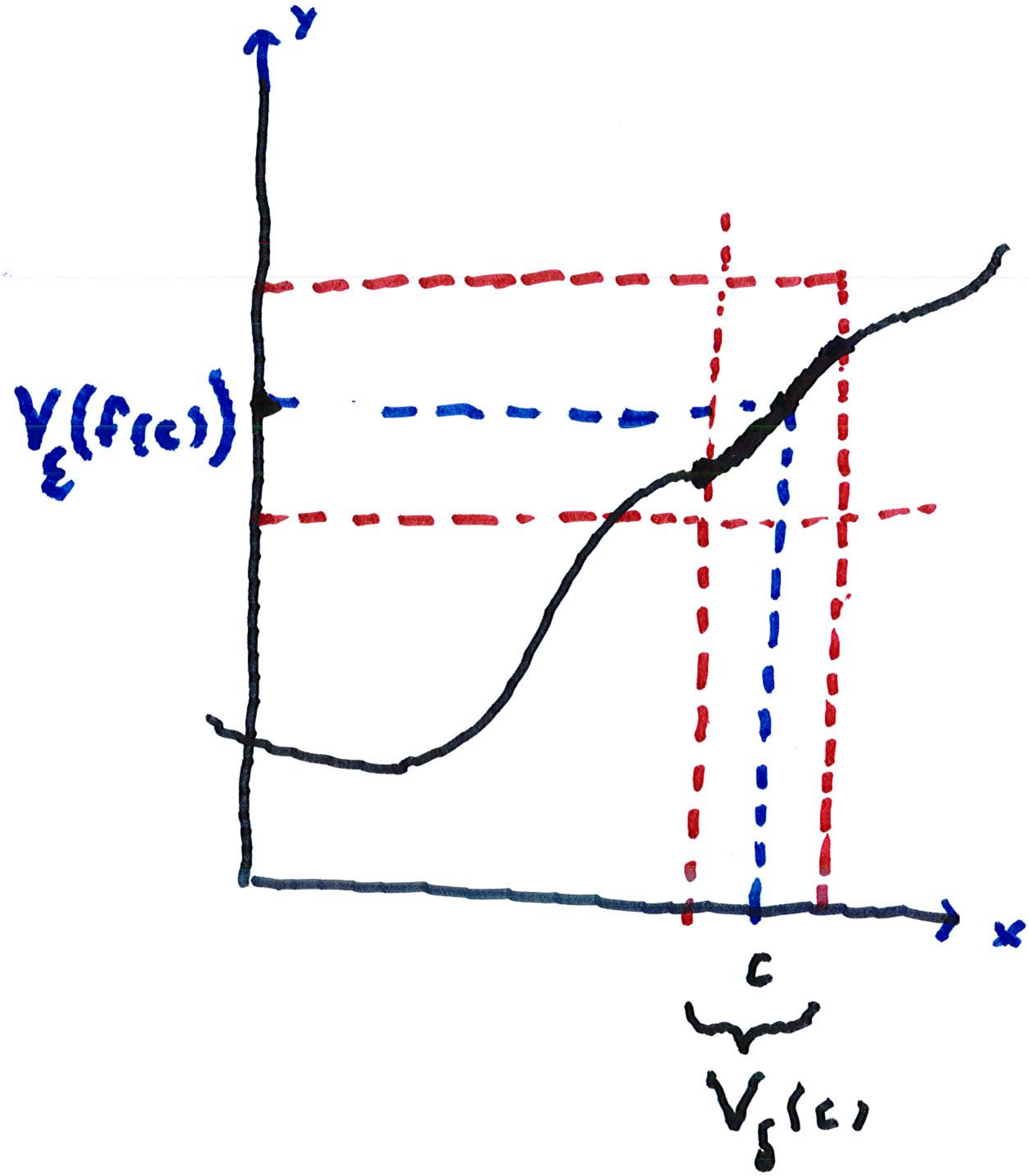
then we can write the

limit as follows :

A function $f: A \to \mathbb{R}$ is continuous at c if:

For every ε-neighborhood $V_\varepsilon(f(c))$, there is a δ-neighborhood $V_\delta(c)$ of c such that if x is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(f(c))$.

$\therefore \; f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$.



The y-values of f above $V_\delta(c)$ lie in $V_\varepsilon(f(c))$.

When f and g are continuous at c then

$$\lim_{x \to c} f(x) = f(c)$$ and

$$\lim_{x \to c} g(x) = g(c).$$ Hence,

1. $$\lim_{x \to c} (f+g) = f(c) + g(c)$$

2. $$\lim_{x \to c} (f-g) = f(c) - g(c)$$

3. $$\lim_{x \to c} (fg) = f(c)g(c)$$

4. $\lim_{x \to c} (bf) = bf(c)$

5. If $g(c) \neq 0$, then

$$\lim_{x \to c} (f/g) = f(c)/g(c)$$

This implies that $f+g$,

$f-g$, fg, bf and f/g are

all continuous at c.

(provided that $g(c) \neq 0$ in 5.)

It follows that any polynomial

and also every rational

function $R(x) = P(x)/Q(x)$

are continuous at every c (except when $Q(x) = 0$

We say a function f defined

on A is continuous

on A if f is continuous

at each $c \in A$.

Composition of Continuous Fcns.

Suppose $f: A \to \mathbb{R}$ is continuous at c

and that $g: B \to \mathbb{R}$ is

continuous at $b = f(c)$.

then we'll show

$(g \circ f)(x) = g(f(x))$ is also

continuous at c, provided

$$f(A) \subseteq B.$$

More precicely:

Thm Let $A, B \subseteq \mathbb{R}$ and

let $f: A \to \mathbb{R}$ and

$g: B \to \mathbb{R}$

be functions such that

$f(A) \subseteq B$. If f is continuous

at a point $c \in A$ and g is

continuous at $b = f(c) \in B$, then

the composition $g \circ f$ is

continuous at c.

Proof: Let W_ε be an ε-neighborhood

of $g(b)$. Since g is continuous

at b, there is a δ-neighborhood

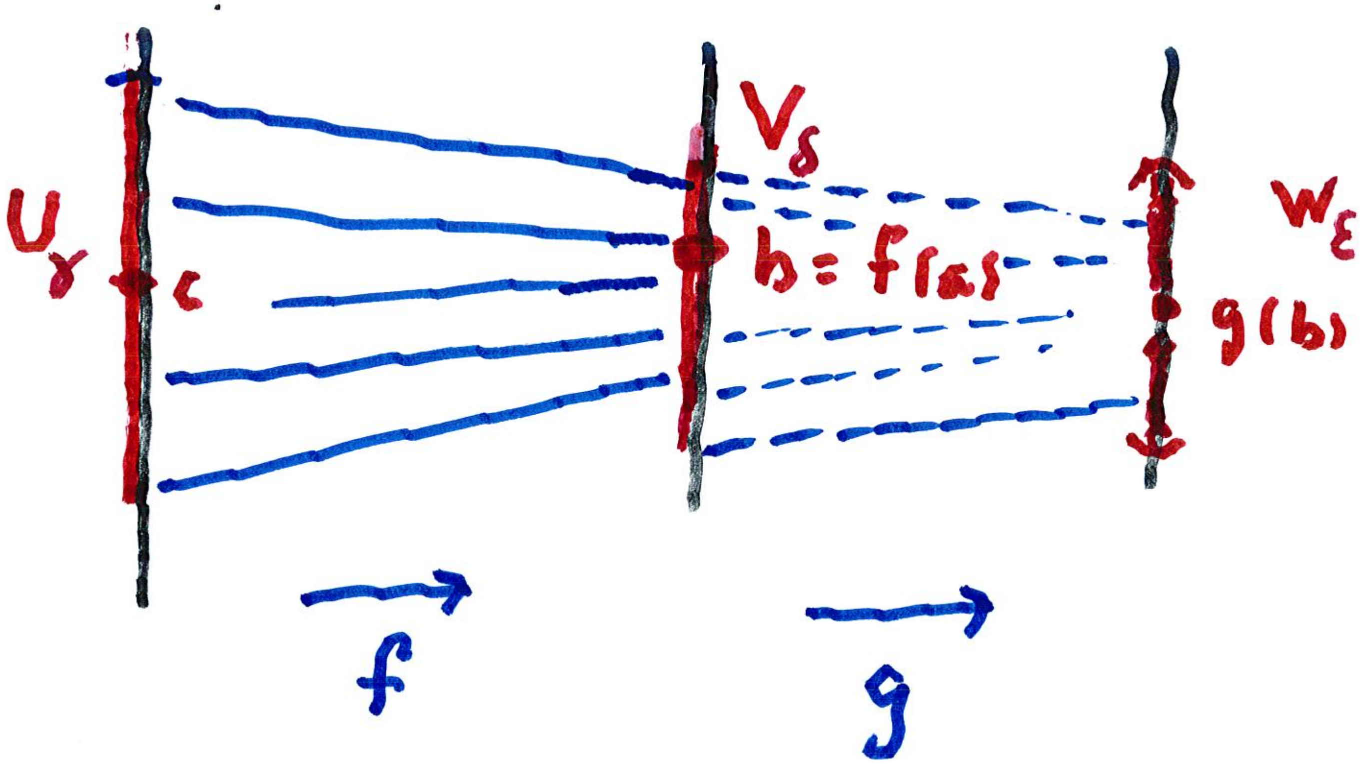
V_δ of $b = f(c)$ such that if $y \in B \cap V_\delta$

then $g(y) \in W_\varepsilon$. Since f is continuous at c, there is a γ-neighborhood U_γ of c such that if $x \in A \cap U_\gamma$ then $f(x) \in V_\delta$. Since $f(A) \subseteq B$, it follows that if $x \in A \cap U_\gamma$, then $f(x) \in B \cap V_\delta$ so that $(g \circ f)(x) = g(f(x)) \in W_\varepsilon$. But since W_ε is an

ε-neighborhood of $g(b)$, this

implies that $g \circ f$ is continuous

at c.



Thm

Let $f: A \to \mathbb{R}$ and let $g: B \to \mathbb{R}$ be continuous on A and B respectively. If $f(A) \subseteq B$, then $g \circ f : A \to \mathbb{R}$ is continuous on A.

Proof: Let c be an arbitrary point in A. Then

f is continuous at c and g is continuous at $b = f(c)$. Hence, the previous theorem implies that $g \circ f$ is continuous at c. Since c is arbitrary, $g \circ f$ is continuous on A.

Ex. Recall from Corollary 2.2.4 that $\big||a|-|b|\big| \leq |a-b|$

Let $g_1(x) = |x|$. We'll show that g_1 is continuous at all points of $\mathbb{R}$. Let $c \in \mathbb{R}$

Let $\varepsilon > 0$, and set $\delta = \varepsilon$.

If $|x-c| < \delta$, then

$$|g_1(x) - g_1(c)| = \big||x| - |c|\big|$$

$\leq |x - c| < \delta = \varepsilon.$

Thus, g_1 is continuous at any $c \in \mathbb{R}$.

Now let f be any function on A that is continuous at $c \in A$. Then $|f(x)|$ is the composition of $g_1(y)$ with f. Hence the previous theorem shows that $|f(x)|$ is continuous at any point c where f is continuous.

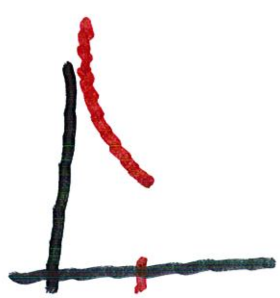
A similar argument shows that $\sqrt{f(x)}$ is continuous at any point c, provided that f is continuous and non-negative in a neighborhood of c.

5.3 Continuous Functions on Intervals

Def'n. A function $f: A \to \mathbb{R}$ is said to be bounded on A if there is a constant $M > 0$ such

$|f(x)| \leq M, \quad$ for all $x \in A$.

Ex $f(x) = \frac{1}{x}$ is not bounded on $(0, 1]$



Thm. Let $I = [a,b]$ and let $f: I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Pf. (by Contradiction)

Suppose f is not bounded.

Then for every integer $n \in N$, there is a point x_n in I such that $|f(x_n)| > n$.

Since I is bounded,

the sequence $X = (x_n)$ is bounded. Hence the Bolzano-Weierstrass Theorem implies there is a subsequence $X' = (x_{n_k})$ of X that converges to a number x. Since I is closed and the elements of X' belong to I, it follows that $x \in I$.

In fact, Theorem 3.2.6 implies that if (y_n) is a convergent sequence and if $a \le y_n \le b$ for all $n \in \mathbb{N}$, then

$$a \le \lim y_n \le b.$$

Then f is continuous at x, so that $(f(x_{n_r}))$ converges to $f(x)$. We then conclude that the convergent sequence $(f(x_{n_r}))$ must be bounded. But this is a contradiction since

$$|f(x_{n_r})| > n_r \geq r, \quad \text{all } r \in \mathbb{N}$$

Hence f is bounded on I.